Functions

- Ordered tuples and Cartesian Product
- Function and Representations
- Types of Functions
- Composition and Inverse of Function
- Numeric Functions

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Types of functions: Review

A function $f : X \mapsto Y$ is **one-to-one** (or **injective**) iff

$$\forall\, x_1, x_2 \in X\ (f(x_1) = f(x_2) \to x_1 = x_2)$$

A function $f : X \mapsto Y$ is **onto** (or **surjective**) iff

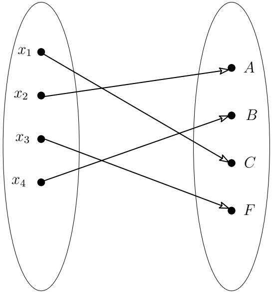
for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$

A function $f : X \mapsto Y$ is **one-to-one correspondence** (or **bijective**) iff

it is **both one-to-one** and **onto**

Inverse of a function

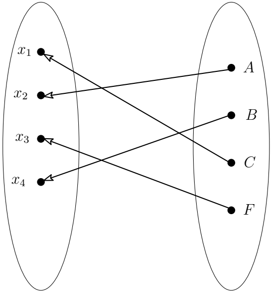
If $f : X \mapsto Y$ is a bijection, then if we reverse the arrows we get a bijection



bijection

onto

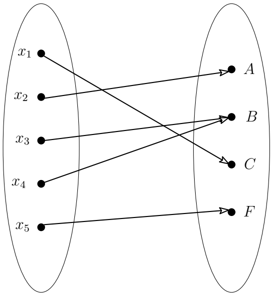
one-to-one



Inverse of a function

If $f : X \mapsto Y$ is a bijection, then if we reverse the arrows we get a bijection

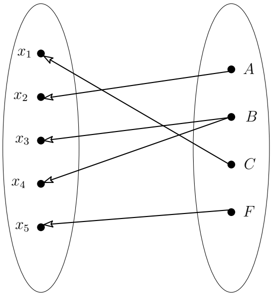
ICP 5-24 Can we say this when f is not one-to-one? a) Yes b) No



Not bijection

onto

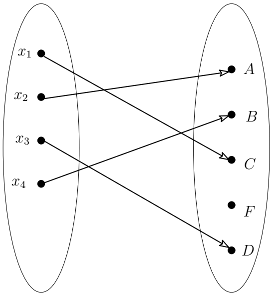
not one-to-one



Inverse of a function

If $f : X \mapsto Y$ is a bijection, then if we reverse the arrows we get a bijection

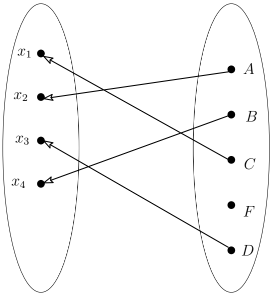
ICP 5-25 Can we say this when f is not onto? a) Yes b) No



Not bijection

not onto

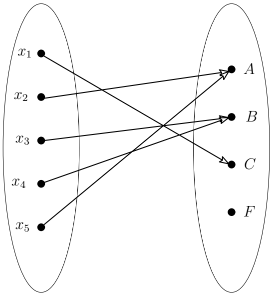
one-to-one



Inverse of a function

If $f : X \mapsto Y$ is a bijection, then if we reverse the arrows we get a bijection

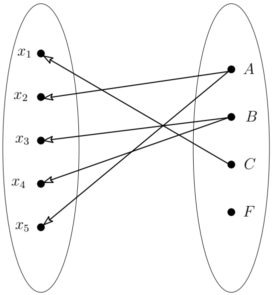
ICP 5-26 Can we say this when f is neither? a) Yes b) No



Not bijection

not onto

not one-to-one



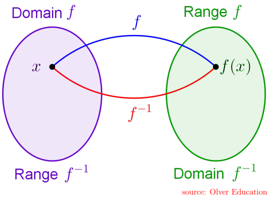
Inverse of function

If $f : X \mapsto Y$ is a bijection, then if we reverse the arrows we get a bijection

If $f : X \mapsto Y$ is a bijection, $f^{-1} : Y \mapsto X$ is the inverse function such that

$$f^{-1}(b) = a, \text{ when } f(a) = b$$

If an inverse exists, then $f(a) = b \leftrightarrow f^{-1}(b) = a$



Inverse of function

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be

$$f(x) = 2x - 3$$

Is f a bijection? What is f^{-1}?

For any $y \in \mathbb{R}$, since ${}^{y+3}/_{2} \in \mathbb{R}$ and $f({}^{y+3}/_{2}) = y$

$\triangleright$ so f is onto

$f(x_1) = f(x_2) \rightarrow 2(x_1) - 3 = 2(x_2) - 3 \rightarrow x_1 = x_2$

$\triangleright$ so f is one-to-one

Hence, f is a bijection

Suppose $f^{-1}(y) = x$

Let $y = 2x - 3$ and solve for x, we get

$$f^{-1}(y) = {}^{y+3}/_{2}$$

Inverse of function

ICP 5-27

Let $f : \mathbb{R}^- \mapsto \mathbb{R}^+$ be

$$f(x) = x^2$$

Is f a bijection? What is f^{-1}?

$\mathbb{R}^- = \{x \in \mathbb{R} \mid x \leq 0\}$ and $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$

$$f^{-1}(y) = -\sqrt{y}$$

Composing functions

The output of one function can be used as input to another function

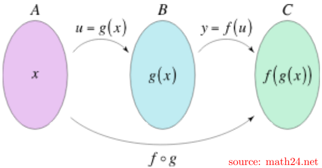
▷ Restriction on range and domain

Let $g : A \mapsto B$ and let $f : B \mapsto C$

The composition of the functions f and g is a function, $(f \circ g) : A \mapsto C$ is

$$(f \circ g)(a) \;=\; f(g(a))$$

- The output of g is input to f
- Range of g (actual outputs) must be a subset of domain of f



source: math24.net

Composition

- Let $f : \mathbb{R} \mapsto \mathbb{R}$ be $f(x) := x^3$
- Let $g : \mathbb{Z} \mapsto \mathbb{Z}$ be $g(x) := x^2$

- $(f \circ g)(x) : \mathbb{Z} \mapsto \mathbb{R}$

$$(f \circ g)(x) = f(g(x)) = f(x^2) = (x^2)^3 = x^6$$

What will be $(g \circ f)$?

Range of inner function must be a subset of domain of outer function

Composition

- Let $f : \mathbb{Z} \mapsto \mathbb{Z}$ be $f(x) := x^3$
- Let $g : \mathbb{Z} \mapsto \mathbb{Z}$ be $g(x) := x^2$

- $(f \circ g)(x) : \mathbb{Z} \mapsto \mathbb{Z}$

$$(f \circ g)(x) \;=\; f(g(x)) \;=\; f(x^2) \;=\; (x^2)^3 \;=\; x^6$$

- $(g \circ f)(x) : \mathbb{Z} \mapsto \mathbb{Z}$

$$(g \circ f)(x) \;=\; g(f(x)) \;=\; g(x^3) \;=\; (x^3)^2 \;=\; x^6$$

Composition

- Let $f : \mathbb{Z} \mapsto \mathbb{Z}$ be $f(x) := x^2$
- Let $g : \mathbb{Z} \mapsto \mathbb{Z}$ be $g(x) := 2x$
- **ICP 5-28** $(f \circ g)(x) : \mathbb{Z} \mapsto \mathbb{Z}$

$$(f \circ g)(x) = f(g(x)) = f(2x) = (2x)^2 = 4x^2$$

- **ICP 5-29** $(g \circ f)(x) : \mathbb{Z} \mapsto \mathbb{Z}$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = 2(x^2) = 2x^2$$

$(f \circ g)(\cdot)$ is not necessarily the same as $(g \circ f)(\cdot)$

Sometimes, the other side of composition may not even be possible

Composing function with itself

A function can be composed with itself

Let $f : \mathbb{Z} \mapsto \mathbb{Z}$ be

$$f(x) = x + 1$$

$$(f \circ f)(x) \ = \ f(f(x)) \ = \ f(x+1) \ = \ (x+1)+1 \ = \ x+2$$

Composing function with itself repeatedly

A function can be composed with itself multiple times

$f : \mathbb{Z} \mapsto \mathbb{Z}$ be

$$f(x) \;=\; x + 1$$

$$(f \circ (f \circ f))(x) \;=\; f(f(f(x))) \;=\; f(f(x+1)) \;=\; f(x+2) \;=\; x + 3$$

Let *father* : *People* $\mapsto$ *People* be defined as

$$\mathit{father}(x) \;=\; \text{father of } x$$

$$\mathit{father}(\mathit{father}(\mathit{father}(x))) \;=\; \text{great grand father of } x$$

Composing function with it's inverse

When a function is invertible (when it's a bijection), then it can be composed with it's inverse

$$\text{If } f = g^{-1} \text{, then } (f \circ g)(x) = f(g(x)) = g^{-1}(g(x)) = x$$

Let $f : \mathbb{Z} \mapsto \mathbb{Z}$ be $f(x) = x + 1$

Let $g : \mathbb{Z} \mapsto \mathbb{Z}$ be $g(x) = x - 1$

$\triangleright$ $f = g^{-1}$

$$f \circ g(x) = f(g(x)) = f(x-1) = (x-1)+1 = x$$

Function inverse and composition: Summary

- If $f : X \mapsto Y$ is a bijection, then if we reverse the arrows we get a bijection too
- If $f : X \mapsto Y$ is a bijection, $f^{-1} : Y \mapsto X$ is the inverse function such that $f^{-1}(b) = a$, when $f(a) = b$
- Let $g : A \mapsto B$ and let $f : B \mapsto C$. The composition of the functions f and g is a function, $(f \circ g) : A \mapsto C$ is $(f \circ g)(a) = f(g(a))$
- Function can be composed with itself (multiple times) and with it's inverse