Functions

- Ordered tuples and Cartesian Product
- Function and Representations
- Types of Functions
- Composition and Inverse of Function
- Numeric Functions

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Functions: Summary

- $f : X \mapsto Y$ maps **each** element of X to **exactly one** element of Y

Let $f : X \mapsto Y$ and let $f(x) = y$

- X is the domain of f
- Y is the codomain of f
- y is the image of x
- x is the pre-image of y
- Range of f: set of images of all elements of X

- Functions can be represented by
  - Listing set of all (pre-image, image) ordered pairs
  - Bipartite Graph
  - Mapping Rule or Algebraic Expression
  - Programming Code

Types of functions: One-to-One

A function $f : X \mapsto Y$ is **one-to-one** (or **injective**) iff

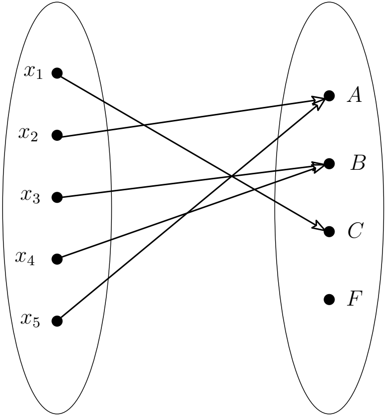
$$\forall\, x_1, x_2 \in X \ \left(f(x_1) = f(x_2) \rightarrow x_1 = x_2 \right)$$

Each element of X is mapped to a unique element of Y

Think of the contrapositive

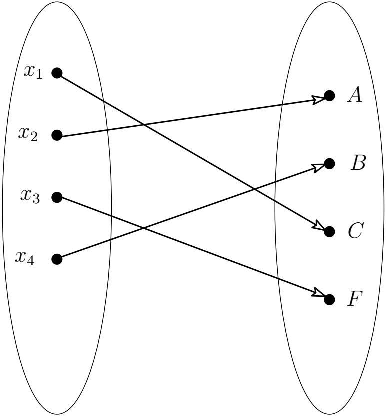
$$\forall\, x_1, x_2 \in X \ \left(x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)\right)$$

Types of functions: One-to-One



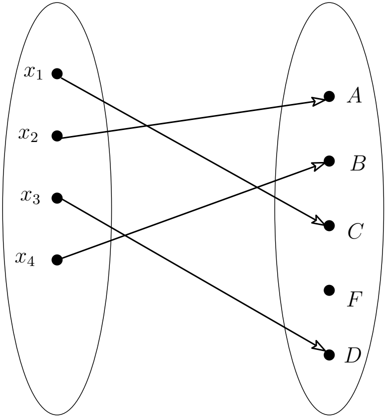
Not one-to-one
$f(x_2) = f(x_5)$

Types of functions: One-to-One



one-to-one

Types of functions: One-to-One



one-to-one

Types of functions: One-to-One

Let $f : \mathbb{Z} \mapsto \mathbb{Z}$ be

$$f(x) = 2x - 3$$

Is f one-to-one?

$$f(x_1) = f(x_2) \rightarrow 2(x_1) - 3 = 2(x_2) - 3 \rightarrow x_1 = x_2$$

Hence, f is one-to-one

Types of functions: One-to-One

Which of the following functions is one-to-one?

ICP 5-9 $f : \mathbb{Z} \mapsto \mathbb{R},\ f(x) = x^2$ a) True b) False

ICP 5-10 $f : \mathbb{R} \mapsto \mathbb{R},\ f(x) = x^3$ a) True b) False

ICP 5-11 $f : \mathbb{Z} \mapsto \mathbb{Z},\ f(x) = 2x$ a) True b) False

ICP 5-12 $f : \mathbb{Z} \mapsto \mathbb{N},\ f(x) = |x|$ a) True b) False

ICP 5-13 f : people $\mapsto$ people,
$f(x) =$ father of x a) True b) False

Types of functions: Onto

A function $f : X \mapsto Y$ is **onto** (or **surjective**) iff

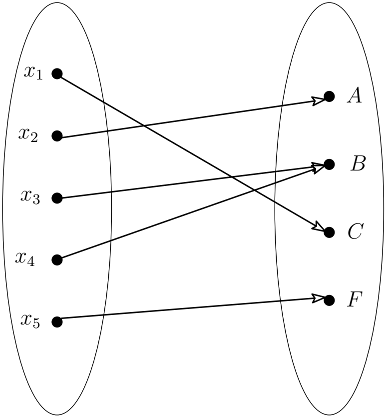
for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$

Each element of Y is assigned to some element of X

$$\forall\, y \in Y\ \exists x \in X\ \ f(x) = y$$

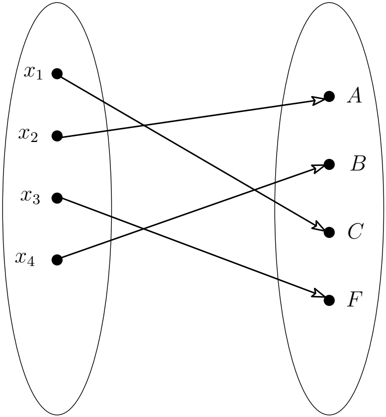
Range = codomain

Types of functions: Onto



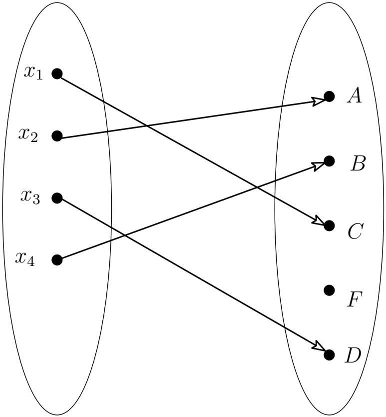
onto

Types of functions: Onto



onto

Types of functions: Onto



not onto

Types of functions: Onto

Which of the following functions is onto?

ICP 5-14 $f : \mathbb{Z} \mapsto \mathbb{R},\ f(x) = x^2$ a) True b) False

ICP 5-15 $f : \mathbb{R} \mapsto \mathbb{R},\ f(x) = x^3$ a) True b) False

ICP 5-16 $f : \mathbb{Z} \mapsto \mathbb{Z},\ f(x) = 2x$ a) True b) False

ICP 5-17 $f : \mathbb{Z} \mapsto \mathbb{N},\ f(x) = |x|$ a) True b) False

ICP 5-18 f : people $\mapsto$ people,
$f(x) =$ father of x a) True b) False

Types of functions: Onto

Let $f : \mathbb{Z} \mapsto \mathbb{Z}$ be

$$f(x) = 2x - 3$$

What is the range of f? Is f onto?

We characterize *range*(f) (a set) from the definition of f and check if it equals codomain

$$\begin{aligned} y \in range(f) &\leftrightarrow \quad y = 2x - 3 \qquad\qquad x \in \mathbb{Z} \\ &\leftrightarrow \quad y = 2(x-2) + 1 \\ &\leftrightarrow \quad y \text{ is odd} \end{aligned}$$

$range(f) \neq codomain(f)$

Hence, f is not onto

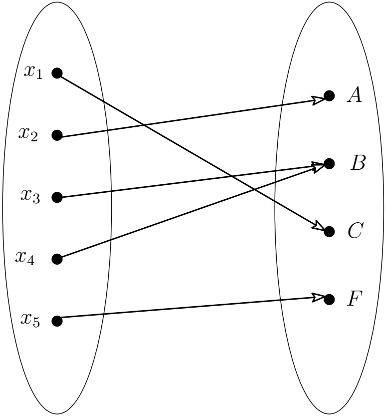
Types of functions: Bijection

A function $f : X \mapsto Y$ is **one-to-one correspondence** (or **bijective**) iff it is **both one-to-one** and **onto**

- Each element of X is mapped to a unique element of Y
- Each element of Y is assigned to some element of X

- if X and Y are finite sets, then $|X| = |Y|$
- **$|$domain$| = |$codomain$|$**

Types of functions: Bijection

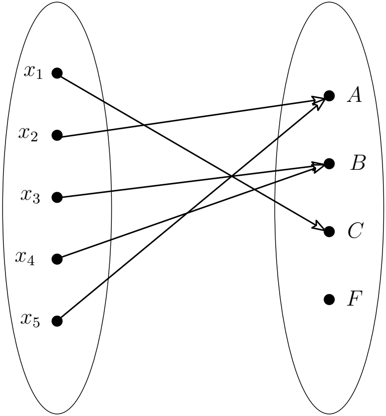


Not bijection

onto

not one-to-one

Types of functions: Bijection



Not bijection

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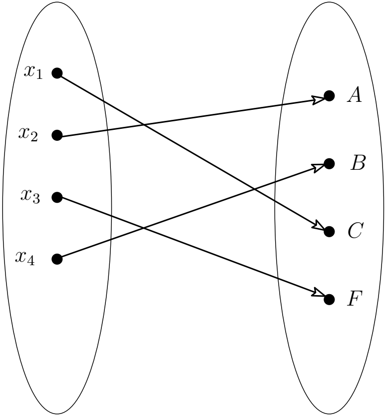
Types of functions: Bijection

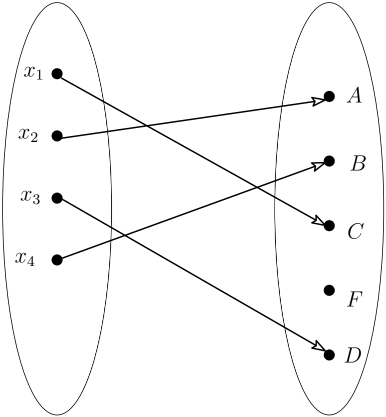


bijection

onto

one-to-one

Types of functions: Bijection



Not bijection

not onto

one-to-one

Types of functions: Bijection

Which of the following functions is a bijection?

ICP 5-19 $f : \mathbb{Z} \mapsto \mathbb{R}$, $f(x) = x^2$ a) True b) False

ICP 5-20 $f : \mathbb{R} \mapsto \mathbb{R}$, $f(x) = x^3$ a) True b) False

ICP 5-21 $f : \mathbb{Z} \mapsto \mathbb{Z}$, $f(x) = 2x$ a) True b) False

ICP 5-22 $f : \mathbb{Z} \mapsto \mathbb{N}$, $f(x) = |x|$ a) True b) False

ICP 5-23 f : people $\mapsto$ people,
$f(x) =$ father of x a) True b) False

Types of functions: Bijection

Let $f : \mathbb{R} \mapsto \mathbb{R}$ be

$$f(x) = 2x - 3$$

Is f a bijection?

- For any $y \in \mathbb{R}$, since $y+3/2 \in \mathbb{R}$ and $f(y+3/2) = y$

  $\triangleright$ so f is onto

- $f(x_1) = f(x_2) \rightarrow 2(x_1) - 3 = 2(x_2) - 3 \rightarrow x_1 = x_2$

  $\triangleright$ so f is one-to-one

- Hence, f is a bijection

Types of functions: Cardinalities

If $f : X \mapsto Y$ is a bijection and X and Y are finite sets, then $|X| = |Y|$

What can we conclude about $|X| \oslash |Y|$,

1. when f is one-to-one?
2. when f is onto?

Types of functions: Summary

A function $f : X \mapsto Y$ is **one-to-one** (or **injective**) iff

$$\forall\, x_1, x_2 \in X\ (f(x_1) = f(x_2) \rightarrow x_1 = x_2)$$

A function $f : X \mapsto Y$ is **onto** (or **surjective**) iff

for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$

A function $f : X \mapsto Y$ is **one-to-one correspondence** (or **bijective**) iff

it is **both one-to-one** and **onto**

If $f : X \mapsto Y$ is a bijection and X and Y are finite sets, then $|X| = |Y|$