Set Theory

- Sets: Definition, Universal Set, Complement, Cardinality
- Subset and Power Set
- Sets Operations
- Set Equality
- Characteristic Vectors: Sets as Bit-Vectors
- Multisets

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Sets Summary

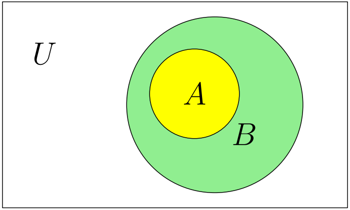
- A set is an ordered collection of objects
- Order and repetition of objects do not matter
- Sets can be described in various ways
- Empty set is a well-defined set with zero objects
- Two sets are equal if and only if they have the same elements
- $\overline{A}$ is the collection of all objects in universal set that are not in A
- Cardinality of A is the number of distinct elements in A

Subset

A is a subset of B if and only if every element of A is an element of B

Denoted by $A \subseteq B$

$$A \subseteq B \quad \text{means} \quad \forall x \, (x \in A \rightarrow x \in B)$$



$A \subseteq B$

When $A \subseteq B$, B is superset of A

Subset

- $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$ ▷ **Natural numbers**
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ ▷ **Integers**
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ ▷ **Positive integers**
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$ ▷ **Rational numbers**

Which of the following is True/False?

ICP 4-6 $\mathbb{N} \subseteq \mathbb{Z}$

a) True

b) False

ICP 4-7 $\mathbb{Z} \subseteq \mathbb{Z}^+$

a) True

b) False

Subset

- $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$ ▷ **Natural numbers**
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ ▷ **Integers**
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ ▷ **Positive integers**
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$ ▷ **Rational numbers**

Which of the following is True/False?

ICP 4-8 $\mathbb{Z}^+ \subseteq \mathbb{N}$

a) True

b) False

ICP 4-9 $\mathbb{Q} \subseteq \mathbb{N}$

a) True

b) False

Subset

- $\mathbb{N} = \{0, 1, 2, 3 \dots\}$ ▷ **Natural numbers**
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ ▷ **Integers**
- $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ ▷ **Positive integers**
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$ ▷ **Rational numbers**

Which of the following is True/False?

ICP 4-10 $\mathbb{Z} \subseteq \mathbb{Q}$

a) True

b) False

ICP 4-11 $\mathbb{Z}^+ \subseteq \mathbb{Q}$

a) True

b) False

Subset

- $\mathbb{N} = \{0, 1, 2, 3 \dots\}$ ▷ **Natural numbers**
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ ▷ **Integers**
- $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$ ▷ **Positive integers**
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$ ▷ **Rational numbers**

Which one of the following is True/False?

ICP 4-12 $\mathbb{Z} \subseteq \mathbb{N}$

a) True

b) False

ICP 4-13 $\mathbb{Z}^+ \subseteq \mathbb{Z}$

a) True

b) False

Empty set is subset of every set

A is a subset of B if and only if every element of A is an element of B

$$A \subseteq B \quad \text{means} \quad \forall x \left(x \in A \rightarrow x \in B\right)$$

$$\forall A \quad \emptyset \subseteq A$$

Need to show that the following is true

$$\forall x \left(x \in \emptyset \rightarrow x \in A\right)$$

$$x \in \emptyset \text{ is always false } \; (\text{ for every } x)$$

thus

$$\left(x \in \emptyset \rightarrow x \in A\right) \text{ is always true}$$

Every set is subset of itself

A is a subset of B if and only if every element of A is an element of B

$$A \subseteq B \quad \text{means} \quad \forall x \, \big(x \in A \rightarrow x \in B\big)$$

$$\forall A \quad A \subseteq A$$

Need to show that the following is true

$$\forall x \, \big(x \in A \rightarrow x \in A\big)$$

$$\big(x \in A \rightarrow x \in A\big) \text{ is always true } \;(\text{ for every } x)$$

Proper Subset

A set A is called a proper subset of B if $A \subseteq B$ but $A \neq B$

Denoted by $A \subset B$ or $A \subsetneq B$

$$A \subset B \quad \text{means} \quad \forall x \, (x \in A \rightarrow x \in B) \wedge \exists x \, (x \in B \wedge x \notin A)$$

Set Equality $A = B$

Two sets A and B are equal if $A \subseteq B$ and $B \subseteq A$

- $A = B$ means $\forall x\, (x \in A \leftrightarrow x \in B)$ ▷ earlier definintion
- $A = B$ means $A \subseteq B$ AND $B \subseteq A$

$$A = B \text{ means } \underbrace{\forall x\, (x \in A \rightarrow x \in B)}_{A \subseteq B} \text{ AND } \underbrace{\forall x\, (x \in B \rightarrow x \in A)}_{B \subseteq A}$$

- Combining the two we get our old definition of $A = B$

Power set

The power set of a set A is the set of all subsets of A

Denoted by $\mathcal{P}(A)$

$A = \{p, q, r\}$

$$\mathcal{P}(A) = \Big\{\emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}, \{p, q, r\}\Big\}$$

- $\emptyset \notin A$ but $\emptyset \subseteq A$, thus $\emptyset \in \mathcal{P}(A)$
- $A \notin A$ but $A \subseteq A$, thus $A \in \mathcal{P}(A)$

Power set

Power set of the empty set, $\mathcal{P}(\emptyset)$

$$\mathcal{P}(\emptyset) = \Big\{\emptyset\Big\} = \Big\{ \{\} \Big\}$$

Power set of the set containing $\emptyset$, $\mathcal{P}\Big(\{\emptyset\}\Big)$

$$\mathcal{P}\Big(\{\emptyset\}\Big) = \Big\{\emptyset, \{\emptyset\}\Big\}$$

Cardinality of Power set

If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$

- $A = \{p, q, r\}$
- $\mathcal{P}(A) = \Big\{\emptyset, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}, \{p, q, r\}\Big\}$
- $|\mathcal{P}(A)| = 2^3 = 8$

ICP 4-14 Let $B = \{p\}$. What is $|\mathcal{P}(B)|$?

ICP 4-15 Let $C = \{\} = \emptyset$. What is $|\mathcal{P}(C)|$?

ICP 4-16 Let $D = \Big\{\{\}\Big\} = \{\emptyset\}$. What is $|\mathcal{P}(D)|$?

Subsets: Summary

- A is a subset of B if and only if every element of A is an element of B
- $A \subseteq B$, A is subset of B, B is superset of A
- Empty set is a subset of every set
- Every set is a subset of itself
- Power Set of A is the set of all subsets of A
- Cardinality of power set of A with $|A| = n$ is 2^n