Set Theory

- Sets: Definition, Universal Set, Complement, Cardinality
- Subset and Power Set
- Sets Operations
- Set Equality
- Characteristic Vectors: Sets as Bit-Vectors
- Multisets

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Set

A set is an unordered collection of objects

- A : $\{1, 2, a, b, Fred, LUMS\}$
- $\mathbb{N}$: $\{0, 1, 2, 3 \ldots\}$
- B : the set of all professors in LUMS
- B : $\{x \mid x \text{ is a professor in LUMS }\}$

Order of elements is not significant

- $\{1, 2, 3\}$ is the same as $\{2, 3, 1\}$

Repetition does not count

- $\{1, 2, 2, 2, 3\}$ is the same as $\{1, 2, 3\}$

Different from arrays in C++/Java (order, same type)

Sets: Notation

Notation:

- Names of sets usually denoted by upper case letters
- Objects in a set are called it's elements/members
- $x \in A$: x is an element of A
- $A \ni x$: A contains x

Sets: Description

- Sets can be described by listing it's elements
  - TAs $:= \{20102222, 20103333, 20104444, 20105555\}$
  - $\mathbb{N} := \{0, 1, 2, 3 \dots\}$
- Sets can be described by an English phrase
  - Set of students in CS-210
  - Set of all professors in LUMS
- Sets can be described by providing a membership predicate
  - Objects for which the predicate is true will be members of the set
  - $B := \{x \,|\, x \text{ is a professor in LUMS}\}$
  - $C := \{x \in \mathbb{N} \,|\, x > 10\}$

Sets: Description

- Sets could have many different equivalent descriptions
- Follows from logical equivalence of membership predicates

ICP 4-1 List 5 elements of each of the following sets

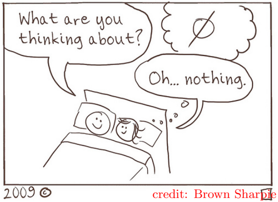
1. $A = \{x \mid x \text{ is even and a perfect square}\}$
2. $B = \{x \mid x + 1 \text{ is a multiple of } 4\}$
3. $C = \{x \text{ rational number} \mid x^2 < 2\}$
4. $X = \{x^2 \mid x \text{ is even}\}$
5. $Y = \{4x - 1 \mid x \in \mathbb{Z}\}$
6. $Z = \{\sqrt{x} \mid x < 2\}$

Standard Numerical Sets

- $\mathbb{N} := \{0, 1, 2, 3 \ldots\}$ ▷ Natural numbers
- $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ ▷ Integers
- $\mathbb{Z}^+ := \{\ldots, 1, 2, 3, \ldots\}$ ▷ Positive integers
- $\mathbb{Q} := \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$ ▷ Rational numbers
- $\mathbb{R} :=$ the set of **Real numbers**
- **int** $:=$ set of integers that can be expressed in 32 bits
- **double** $:=$ the set of real numbers that can be expressed in 64 bits

Some other sets

- Empty Set
  - $A = \emptyset = \{\ \}$
- Family/Collection of sets: Set of sets
  - $A = \Big\{\{x, y\}, \{y, z\}, \{z\}\Big\}$
- Set containing sets
  - $A = \Big\{\{p, q\}, x, \{x\}, \{y\}\Big\}$



credit: Brown Sharpie

Set Equality

Two sets are equal if and only if they have the same elements

$$A = B \quad \text{means} \quad \forall x \, (x \in A \leftrightarrow x \in B)$$

ICP 4-2 $A = \{3, 1, -3, 9\}$ $\qquad$ $B = \{-3, 1, 3, 9\}$

Is $A = B$? why?

Set Equality

Two sets are equal if and only if they have the same elements

$$A = B \quad \text{means} \quad \forall x \, (x \in A \leftrightarrow x \in B)$$

ICP 4-3 $A = \{p, q, r, s\}$ $B = \{r, q, s\}$

Is $A = B$? why?

Set Equality

Two sets are equal if and only if they have the same elements

$$A = B \quad \text{means} \quad \forall x\,(x \in A \leftrightarrow x \in B)$$

ICP 4-3 $A = \{p, q, r\}$ $\qquad B = \{p, r, q, p\}$

Is $A = B$? why?

Set Complement

The complement of a set A contains those elements that are not in A

- $\mathbb{Q} = \{ {}^{p}/_{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0 \}$ **(rationals)**
- **Irrational numbers**: non-rational number
- $\{ \sqrt{2}, \ \pi, \ \ e, \ $ cat, dog, calculus $\ldots \}$
- Universal Set ▷ recall Universe of Discourse

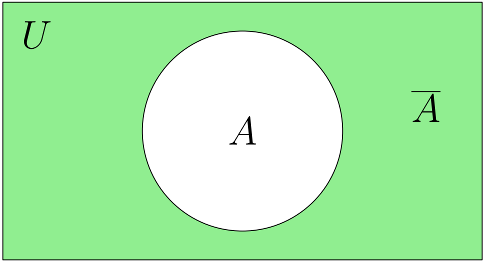
The complement of a set A contains those elements (of the universal set) that are not in A

Set Complement

The complement of a set A consists of all elements of U that are not in A

Denoted by $\overline{A}$

$$\overline{A} := \{x \in U \mid x \notin A\}$$



$\overline{A}$

Puzzle

In a town, there is a male barber who shaves all the men and only those men who do not shave themselves

Does the barber shave himself?

- Let $M = \{\text{men who shave themselves}\}$
- M is the set of people that the barber b does not shave
- If the barber b shaves himself, then $b \in M$
- Then barber must not shave himself, i.e. $b \notin M$
- But barber shaves all those not in M, so $b \in M$
- $b \in M$ is neither true nor false

Universal Set

Russell's Paradox

Let S be the set that contains all sets, which do not contain themselves

Can this set exist?

- Let say S exists. Then either $S \in S$ or $S \notin S$
- If $S \in S$, since all elements of S do not contain themselves, thus S does not contain S, i.e. $S \notin S$
- If $S \notin S$, since S contains all those sets that do not contain themselves, so S should contain S, i.e. $S \in S$
- In both cases the assumption is contradicted

Cardinality of finite sets

For a finite set A it's cardinality is the number of distinct elements in A

Denoted by $|A|$

$A = \{\text{Even integers among the first 100 positive integers}\}$

ICP 4-4 $|A| = ?$

$B = \{\text{Positive factors of 16}\}$

ICP 4-5 $|B| = ?$

Sets Summary

- A set is an ordered collection of objects
- Order and repetition of objects do not matter
- Sets can be described in various ways
- Empty set is a well-defined set with zero objects
- Two sets are equal if and only if they have the same elements
- $\overline{A}$ is the collection of all objects in universal set that are not in A
- Cardinality of A is the number of distinct elements in A