Predicate Logic

- Predicates and Propositional Functions
- Universal and Existential Quantifiers
- Negating Quantified Statements
- Nested Quantified Expressions

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Quantified Expression: Recap

- Propositional function becomes proposition when specific value is given to variable
- Quantifiers make it proposition for a range of values

- **Universal Quantifier:** $\forall$

  $\forall x\ P(x) \quad := \quad P(x)$ (is true) for all values of x in the UoD

Proposition $\forall x\ P(x)$ is **True** iff for every x in UoD, $P(x)$ is **True**

- **Existential Quantifier:** $\exists$

  $\exists x\ P(x) \quad := \quad P(x)$ (is true) for some value(s) of x in the UoD

Proposition $\exists x\ P(x)$ is **True** iff for at least one x in UoD, $P(x)$ is **True**

Nested Quantified Expressions

Predicates can have more than one variables

- $P(x, y)$: x teaches course y in LUMS
- $Q(x, y, z)$: x is an instructor of course y in university z
- $S(x, y, z)$: $x + y = z$

$P(Pythogarus, CS210) \; = ?$

$P(Pythogarus, y) \; = ?$

$P(x, CS210) \; = ?$

Each variable needs to be given value or quantified to make the predicate a proposition

Nested Quantified Expressions: Binding Variables

Predicates can have more than one variables

Each variable needs to be given value or quantified to make the predicate a proposition

- $\forall x\ P(x, y)$ is not a proposition

  ▷ The variable x is *'bound'*, while y is *'free'*

- $\forall x\ \forall y\ P(x, y)$ is a proposition

  ▷ both variables are bound

- $\forall x\ \exists y\ \forall z\ Q(x, y, z)$ is a proposition

  ▷ all variables are bound

Nested Quantified Expressions: Binding Variables

$$\forall x, \forall y \; P(x,y) \;\; \overset{?}{\equiv} \;\; \forall x \; P(x,y) \wedge \forall y \; P(x,y)$$

LHS: Both x and y are bound

RHS: x is bound in the first predicate and y is free

RHS: y is bound in the second predicate and x is free

RHS is not even a proposition

Nested Quantified Expressions: Examples

Translate these statements to logical expressions with quantifiers

The sum of two positive integers is always positive

$$\forall x \forall y \ \big((x > 0) \wedge (y > 0)\big) \rightarrow (x + y > 0)$$

Every real number except zero has a multiplicative inverse

$$\forall x \exists y \ (x \neq 0) \rightarrow (xy = 1)$$

If a person is female and is a parent, then this person is someone's mother

$$\forall x \exists y \ \big(F(x) \wedge P(x)\big) \rightarrow M(x, y)$$

Everyone has exactly one best friend

$$\forall x \exists y \forall z \ (B(x, y) \wedge (y \neq z)) \rightarrow \neg B(x, z)$$

Nested Quantified Expressions: Examples

Translate the following logical expressions into English statements

$\forall x \exists y \; F(x, y) \land x \neq y$

Everyone has at least one friend

$\exists y \forall x \; F(x, y) \land x \neq y$

There is at least one person who is friend of everyone

Are these statements logically equivalent?

Nested Quantified Expressions: Examples

Translate the logical expressions into English statements

$\forall x \exists y \, (x \neq 0) \rightarrow (xy = 1)$

Every real number except zero has a multiplicative inverse

$\exists y \forall x \, (x \neq 0) \rightarrow (xy = 1)$

Some real number is multiplicative inverse of every non-zero real number

Are these statements logically equivalent?

Nested Quantified Expressions: Order is important

Order of quantifiers is extremely important

$$\forall x \exists y\ P(x,y) \quad \textbf{is not the same as} \quad \exists y \forall x\ P(x,y)$$

$$P(x,y) : x \text{ teaches course } y \text{ in LUMS}$$

- $\forall x\ \exists y\ P(x,y)$: For every instructor, there is a course that (s)he teaches
- $\exists y\ \forall x\ P(x,y)$: There is a course that every instructor teaches
- $\exists x\ \forall y\ P(x,y)$: There is an instructor who teaches all the courses
- $\forall y\ \exists x\ P(x,y)$: For every course, there is an instructor who teaches it

Nested Quantified Expressions: Order is important

Order of quantifier is extremely important

$$\forall x \exists y\ P(x, y) \quad \textbf{is not the same as} \quad \exists y \forall x\ P(x, y)$$

$Q(x, y, z)$: x is an instructor of course y in university z

- $\forall z\ \forall y\ \exists x\ Q(x, y, z)$: In every uni. for every course there is an instructor
- $\forall z\ \exists x\ \forall y\ Q(x, y, z)$: In every uni. there is an instructor for all courses
- $\exists x\ \forall z\ \forall y\ Q(x, y, z)$: There is an instructor for every course in every uni.

Nested Quantified Expressions: Order is important

1				
		1	1	
1				
				1
				1

Table: **1**

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

Table: **2**

- $\forall$ row x $\exists$ column y $A[x][y] = 1$
- Every row has at least one 1
- Both tables satisfy this

- $\exists$ column y $\forall$ row x $A[x][y] = 1$
- There is a column with all 1's
- Table **1** does not satisfy this
- Table **2** satisfy this

Nested Quantified Expressions: Order is important

1				
		1	1	
1				
				1
				1

Table: **1**

1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1
1	1	1	1	1

Table: **2**

- $\forall$ row x $\forall$ column y $A[x][y] = 1$
- Every cell has a 1
- Table **1** does not satisfy this
- Table **2** satisfy this

- $\forall$ column y $\forall$ row x $A[x][y] = 1$
- Every cell has a 1
- Table **1** does not satisfy this
- Table **2** satisfy this

Nested Quantified Expressions: Order of Quantifiers

1				
		1	1	
1				
				1
				1

Table: **3**

		1		

Table: **4**

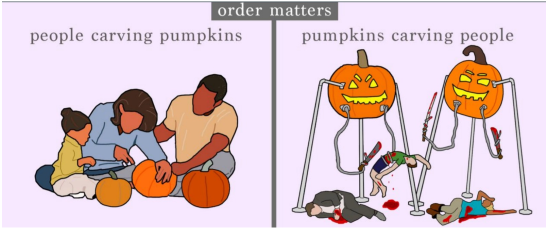
- $\exists$ row x $\exists$ column y $A[x][y] = 1$
- There is a 1 in the table
- Both tables satisfy this

- $\exists$ column x $\exists$ row y $A[x][y] = 1$
- There is a 1 in the table
- Both tables satisfy this

Nested Quantified Expressions

Order of quantifier is extremely important

$$\forall x \exists y\ P(x,y) \quad \textbf{is not the same as} \quad \exists y \forall x\ P(x,y)$$



source: http://wtkrieger.faculty.noctrl.edu/csc230-winter2019/etc/

If both quantifiers are the same then order does not matter

$$\forall x \forall y\ P(x,y) \equiv \forall y \forall x\ P(x,y) \quad \text{and} \quad \exists x \exists y\ P(x,y) \equiv \exists y \exists x\ P(x,y)$$

Nested Quantified Expressions: Examples

Goldbach conjecture (1742)

Every even integer greater than 2 can be written as sum of two primes

$$\forall n \in \mathbb{N}, ((n > 2) \wedge (n \text{ even})) \rightarrow (\exists p, q \in \mathbb{P}, n = p + q)$$

Vinogradov (1937)

Every sufficiently large odd number is the sum of three primes

$$\exists K \in \mathbb{N}, \forall n \geq K, ((n \text{ odd}) \rightarrow \exists p, q, r \in \mathbb{P}, n = p + q + r)$$

Truth Values of Nested Quantified Expressions

Statement	*When True?*	*When False?*
$\forall x \forall y\ P(x,y)$ $\forall y \forall x\ P(x,y)$	$P(x,y)$ is true for every pair x,y	There is a pair x,y for which $P(x,y)$ is false
$\forall x \exists y\ P(x,y)$	For every x, there is a y for which $P(x,y)$ is true	There is an x such that $P(x,y)$ is false for every y
$\exists x \forall y\ P(x,y)$	There is an x for which $P(x,y)$ is true for every y	For every x there is a y for which $P(x,y)$ is false
$\exists x \exists y\ P(x,y)$ $\exists y \exists x\ P(x,y)$	There is a pair x,y for which $P(x,y)$ is true	$P(x,y)$ is false for every pair x,y

Nested Quantified Expressions

Express each of these system specifications using predicates with UoD's, quantifiers, and logical connectives

ICP 2-27 At least one console must be accessible during every fault condition

$A(x, y)$: Console x is accessible during fault condition y

$\forall y \exists x\ A(x, y)$

Nested Quantified Expressions

Express each of these system specifications using predicates with UoD's, quantifiers, and logical connectives

ICP 2-28 The e-mail address of every user can be retrieved whenever the archive contains at least one message sent by every user on the system

$R(y)$: Email of user y can be retrieved

$M(x, y)$: Message x has been sent by user y

$\forall y \forall z \exists x \; M(x, z) \rightarrow R(y)$

Nested Quantified Expressions

Express each of these system specifications using predicates with UoD's, quantifiers, and logical connectives

ICP 2-29 For every security breach there is at least one mechanism that can detect that breach if and only if there is a process that has not been compromised

$D(x, y)$: Mechanism x can detect breach y

$C(z)$: Process z has been compromised

$\forall y \exists x \exists z\ D(x, y) \iff \neg C(z)$

Nested Quantified Expressions

Express each of these system specifications using predicates with UoD's, quantifiers, and logical connectives

ICP 2-30 There are at least two paths connecting every two distinct endpoints on the network

$C(p, x, y)$: Path p connects endpoint x and y

$\forall x \forall y \neq x \, \exists p_1 \exists p_2 \neq p_1 \; C(p_1, x, y) \wedge C(p_2, x, y)$

Nested Quantified Expressions

Translate these statements expressing some mathematical fact into English

Let the Universe of discourse for all variables be real numbers

ICP 2-32 $\forall x \exists y \, (x + y = 0)$

Every real number has an additive inverse

ICP 2-33 $\exists x \forall y \, (x + y = y)$

There exists an additive identity

ICP 2-34 $\forall x \exists y \, \big((x \neq 0) \rightarrow (xy = 1)\big)$

Every non-zero real number has a multiplicative inverse

ICP 2-35 $\exists x \forall y \, (xy = y)$

There exists a multiplicative identity

Negating Nested Quantified Expressions

Recall $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$ $\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$

Negate nested quantified statements using iterative applications of negating (singly) quantified statements

$$\neg\, \forall x\, \exists y\, P(x,y) \equiv \exists x\, \neg\, \exists y\, P(x,y) \equiv \exists x\, \forall y\, \neg\, P(x,y)$$

$$\neg\, \exists x\, \forall y\, P(x,y) \equiv \forall x\, \neg\, \forall y\, P(x,y) \equiv \forall x\, \exists y\, \neg\, P(x,y)$$

$$\neg\, \forall x\, \forall y\, P(x,y) \equiv \exists x\, \neg\, \forall y\, P(x,y) \equiv \exists x\, \exists y\, \neg\, P(x,y)$$

$$\neg\, \exists x\, \exists y\, P(x,y) \equiv \forall x\, \neg\, \exists y\, P(x,y) \equiv \forall x\, \forall y\, \neg\, P(x,y)$$

Negating Nested Quantified Expressions

$P(x, y)$: x teaches course y in LUMS

Negate the following statement in plain English

For every course, there is an instructor teaching it

Next translate it into a quantified expression

Negate the quantified expression

Translate the negated quantified expression into plain English and compare with the one you got earlier

Nested Quantified Expressions: Summary

- All variables in multivariable predicates need to be bound
- Each variable can be quantified differently
- Order of different quantifiers is extremely important
- For same quantifiers order does not matter
- It is tricky to translate English statements to nested quantified statement and vice-versa
- Requires extensive practice
- Negating nested quantified statement is just nested applications negating quantified statements