Predicate Logic

- Predicates and Propositional Functions
- Universal and Existential Quantifiers
- Negating Quantified Statements
- Nested Quantified Expressions

Imdad ullah Khan

Quantified Expression: Recap

- Propositional function becomes proposition when specific value is given to variable
- Quantifiers make it proposition for a range of values

- **Universal Quantifier:** $\forall$
  - $\forall x \, P(x) \quad := \quad P(x)$ (is true) for all values of x in the UoD

Proposition $\forall x \, P(x)$ is **True** iff for every x in UoD, $P(x)$ is **True**

- **Existential Quantifier:** $\exists$
  - $\exists x \, P(x) \quad := \quad P(x)$ (is true) for some value(s) of x in the UoD

Proposition $\exists x \, P(x)$ is **True** iff for at least one x in UoD, $P(x)$ is **True**

Negating Quantified Expressions

A quantified predicate is a proposition

Hence, it's negation is a well defined proposition

Which is **True** when the quantified predicateis **False** and vice-versa

Negating Quantified Expressions

"Every student in this class has taken Calculus"

- $C(x)$: x has taken Calculus
- UoD: Students in this class
- $\forall x \; C(x)$

Negation:

- *"It is not the case that every student in this class has taken Calculus"*
- *"There is a student in this class who has Not taken Calculus"*

Negating Quantified Expressions

"Every student in this class has taken Calculus"

- $C(x)$: x has taken Calculus
- UoD: Students in this class
- $\forall x \; C(x)$

Negation:

- *"It is not the case that every student in this class has taken Calculus"*
- $\underbrace{\text{There is a student in this class}}_{\exists x}\ \underbrace{\text{who has Not taken Calculus.}}_{\neg C(x)}$
- $\exists x \; \neg C(x)$

In general, $\neg \forall x \; P(x) \;\equiv\; \exists x \; \neg P(x)$

Negating Quantified Expressions

"Every student in this class has taken Calculus"

- $C(x)$: x has taken Calculus
- $D(x)$: x is taking DM (x is in this class)
- UoD: Students in LUMS
- $\forall x\,(D(x) \to C(x))$

Negation:

- *"There is a student in this class who has Not taken Calculus"*

$$\neg\forall x\,(D(x) \to C(x)) \;\equiv\; \exists x\,\neg(D(x) \to C(x))$$

$$\equiv\; \exists x\,\neg(\neg D(x) \vee C(x)) \;\equiv\; \exists x\,(D(x) \wedge \neg C(x))$$

The last two equivalences follow from Implication and DeMorgan's Law

Negating Quantified Expressions

"Some student in this class has taken Chemistry"

- $C(x)$: x has taken Chemistry
- UoD: Students in this class
- $\exists x \, C(x)$

Negation:

- It is not the case that there is a student in this class who has taken Chemistry
- $\underbrace{\text{Every student in this class}}_{\forall x} \; \underbrace{\text{has Not taken Chemistry}}_{\neg C(x)}$

In general, $\neg \exists x \, P(x) \equiv \forall x \, \neg P(x)$

Negating Quantified Expressions

"Some student in this class has taken Chemistry"

- $C(x)$: x has taken Chemistry
- $D(x)$: x is taking DM (x is in this class)
- UoD: Students in LUMS
- $\exists x \, (C(x) \wedge D(x))$

Negation:

- *"It is not the case that some student is this class and has taken chemistry"*
- $\underbrace{\text{Every student}}_{\forall x} \; \underbrace{\text{who is this class has not taken Chemistry}}_{\neg(C(x) \wedge D(x))}$

$$\neg \exists x \, (C(x) \wedge D(x)) \equiv \forall x \, \neg(C(x) \wedge D(x)) \equiv \forall x \, (\neg D(x) \vee \neg C(x))$$

Negating Quantified Expressions

DeMorgan's Laws: **1.** $\neg(p \vee q) \equiv \neg p \wedge \neg q$ **2.** $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Universal Quantifier: $\forall$

- $\forall x\ P(x) \quad := \quad P(x)$ (is true) for all values of x in the UoD
- $\forall x\ P(x) :\simeq P(1) \wedge P(2) \wedge P(3) \wedge \ldots \wedge P(n)$ assuming UoD is $\{1 \ldots n\}$
- $\neg\forall x\ P(x) :\simeq \neg[P(1) \wedge \ldots \wedge P(n)] = \neg P(1) \vee \ldots \vee \neg P(n) = \exists x\ \neg P(x)$

Existential Quantifier: $\exists$

- $\exists x\ P(x) \quad := \quad P(x)$ (is true) for some value(s) of x in the UoD
- $\exists x\ P(x) :\simeq P(1) \vee P(2) \vee P(3) \vee \ldots \vee P(n)$ assuming UoD is $\{1 \ldots n\}$
- $\neg\exists x\ P(x) :\simeq \neg[P(1) \vee \ldots \vee P(n)] = \neg P(1) \wedge \ldots \wedge \neg P(n) = \forall x\ \neg P(x)$

$$\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$$

$$\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$$

Truth Values of Quantified Expressions

Statement	*When True?*	*When False?*
$\forall x\ P(x)$	$P(x)$ is true for every x	There is an x for which $P(x)$ is false
$\exists x\ P(x)$	There is an x for which $P(x)$ is true	$P(x)$ is false for every x
$\neg\exists x\ P(x)$	$P(x)$ is false for every x	There is an x for which $P(x)$ is true
$\neg\forall x\ P(x)$	There is an x for which $P(x)$ is false	$P(x)$ is true for every x

Example and Counter Example

All people are smart

- $P(x)$: x is smart. $\forall x\ P(x)$ $\triangleright$ **False**
- **Counter Example:** xxxxxxxxxxxxxxxxxx

Some people are stupid

- $Q(x)$: x is stupid. $\exists x\ Q(x)$ $\triangleright$ **True**
- $\exists x\ \neg P(x)$
- **Example:** xxxxxxxxxxxxxxxxxx

Example and Counter Example

Example and Counter Example

ICP 2-24

Which of the following is equivalent to $\neg \forall x \; P(x)$?

1. $\neg \exists x \; P(x)$
2. $\neg \forall x \; \neg P(x)$
3. $\forall x \; \neg P(x)$
4. $\exists x \; \neg P(x)$

Example and Counter Example

ICP 2-25

Which of the following is equivalent to $\neg \exists x \; P(x)$?

1. $\neg \forall x \; P(x)$
2. $\neg \exists x \; \neg P(x)$
3. $\exists x \; \neg P(x)$
4. $\forall x \; \neg P(x)$

Negating Quantified Expressions

ICP 2-26 Perform the following tasks on the statement

All students in CS 410 are CS majors.

1. Negate it in English. Do not use the phrase "It is not the case that"
   At least one student in CS 410 is **NOT** a CS major
2. Express it using quantifiers (define appropriate predicates and UoD)
   UoD: CS410 students, $C(x) : x$ is CS major $\forall x \; C(x)$
3. Negate the quantified expression, with no $\neg$ symbol left of a quantifier
   $\neg \forall x \; C(x) \equiv \exists x \neg C(x)$
4. Translate the negation into simple English
   There is a CS 410 student who is not CS major
5. Compare this negation with the earlier one
   They are equivalent

Negating Quantified Expressions

Perform the following tasks on each of these statements

- There is a chemistry major in CS 210.
- Every computer scientist does programming.
- No chemistry major knows Python.

1. Negate it in English. Do not use the phrase "It is not the case that"
2. Express it using quantifiers (define appropriate predicates and UoD)
3. Negate the quantified expression, with no $\neg$ symbol left of a quantifier
4. Translate the negation into simple English
5. Compare this negation with the earlier one

Negating Quantified Expressions: Summary

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$

$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

They follows from DeMorgan's law (when UoD are finite)