Predicate Logic

- Predicates and Propositional Functions
- Universal and Existential Quantifiers
- Negating Quantified Statements
- Nested Quantified Expressions

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Propositional function: Recap

$$\underbrace{x}_{subject}\ \underbrace{\text{is greater than 3}}_{predicate}$$

- Denoted by $P(x)$
- P denotes the predicate "is greater than 3"
- x is the variable (subject or argument)
- $P(x)$ is the value of propositional function P at x
- Takes one or more arguments (like multivariable functions)
- Yields the value **true (T)** or **false (F)** for the subject(s)
- Universe of discourse is set of possible values for variables
- Each variable may have a different universe of discourse

Quantifiers

Propositional function becomes proposition when value is given to variable

Quantifiers make it proposition for a range of values

English quantifiers: *all, some, none, many, few, at least one*

We use two quantifiers

1. The universal quantifier
2. The existential quantifier

Universal Quantifier

The universal quantification of $P(x)$ is the proposition

$P(x)$ (is true) for **all** values of x (in the UoD)

- Notation: $\forall x \; P(x)$
- Pronounced as: "For all x, $P(x)$ (is true)

$\forall x \; P(x)$ is *'like'* AND ($\wedge$) over the entire UoD (finite)

$\forall x \; P(x) :\simeq P(1) \wedge P(2) \wedge P(3) \wedge \ldots \wedge P(n)$ ▷ assuming UoD is $\{1 \ldots n\}$

The proposition $\forall x \; P(x)$ is **True** iff for every x in UoD $P(x)$ is **True**

Universal Quantifier

$P(x) : x + 3 > x + 1$ $\triangleright$ UoD: Real numbers

$$\forall x \; P(x)$$

$P(x)$ is true for every real numbers (every element of UoD)

Hence, $\forall x \; P(x)$ is **True**

Universal Quantifier

ICP 2-20

$$\forall x \; x^2 < x$$

$\triangleright$ UoD: Positive Integers

$x^2 < x$ is false for every positive integer

True/**False** ?

Universal Quantifier

ICP 2-21

$$\forall x\ x^2 > x$$

▷ UoD: Nonpositive Integers $= \{\ldots, -3, -2, -1, 0\}$

True/False ?

Careful: $x^2 > x$ is false for 0, which is in UoD

Universal Quantifier

$$\forall x \; x^2 \geq x$$

▷ UoD: Positive Integers

True/False ?

Universal Quantifier

Euler Conjecture

$$\forall\ a, b, c, d \qquad a^4 + b^4 + c^4 \neq d^4$$

▷ UoD: Positive Integers

False: Because Noam Elkies (1987) found one counter example

$a = 2682440$, $b = 15365639$, $c = 18796760$, $d = 20615673$ is a solution

Universal Quantifier

Policy: Calculus is prerequisite of Discrete Mathematics

Translate *"Every student in this class has taken Calculus"*

- $C(x)$: x has taken Calculus
- UoD: Students in this class
- $\forall x\ C(x)$

Universal Quantifier

Translate *"Every student in this class has taken Calculus"*

- $C(x)$: x has taken Calculus
- $D(x)$: x is taking DM (x is in this class)
- UoD: Students in LUMS
- $\forall x\,(C(x) \wedge D(x))$ Wrong
  - ▷ Every student in LUMS has taken Calculus AND is taking DM
- $\forall x\,(D(x) \rightarrow C(x))$

- Translation changes with UoD, so carefully select one
- Try to re-translate to English and see if makes sense

Existential Quantifier

The existential quantification of $P(x)$ is the proposition

$P(x)$ (is true) for some value(s) of x in the UoD

- Notation: $\exists x\ P(x)$
- Pronounced as: "There exists an x, such that $P(x)$ (is true)
- $\exists x\ P(x)$ is *'like'* OR ($\vee$) over the entire UoD (finite)
- $\exists x\ P(x) :\simeq P(1) \vee P(2) \vee P(3) \vee \ldots \vee P(n)$ ▷ assuming UoD is $\{1 \ldots n\}$

Proposition $\exists x\ P(x)$ is **True** iff for at least one x in UoD, $P(x)$ is **True**

Existential Quantifier

$P(x) : x^2 \leq x$ ▷ UoD: Integers

$$\exists x \; P(x)$$

- $P(x)$ is **True** for 0 and 1, which are in the UoD
- Hence, $\exists x \; P(x)$ is **True**

Existential Quantifier

ICP 2-22

$P(x) : x = x + 1$ ▷ UoD: Real numbers

$$\exists x \; P(x)$$

- $P(x)$ is False for every real number
- $P(x)$ is Not True for any element in UoD
- Is $\exists x \; P(x)$ **True/False** ?

Existential Quantifier

ICP 2-23

$$\exists x \; \sqrt{x} < 4$$

$\triangleright$ UoD: $\{10, 12, 14, 16, 18, 20\}$

- $P(x)$ is True for $10, 12, 14$
- Is $\exists x \; \sqrt{x} < 4$ **True/False** ?

Existential Quantifier

Translate: *"Some student in this class has taken Chemistry"*

- $C(x)$: x has taken Chemistry
- UoD: Students in this class
- $\exists x\ C(x)$

Existential Quantifier

Translate: *"Some student in this class has taken Chemistry"*

- $C(x)$: x has taken Chemistry
- $D(x)$: x is taking DM (x is in this class)
- UoD: Students in LUMS
- $\exists x\ (C(x) \wedge D(x))$

Quantified Expression: Summary

- Propositional function becomes proposition when specific value is given to variable
- Quantifiers make it proposition for a range of values
- **Universal Quantifier:** $\forall$
  - $\forall x\ P(x) \quad := \quad P(x)$ (is true) for all values of x in the UoD
  - $\forall x\ P(x) :\simeq P(1) \wedge P(2) \wedge P(3) \wedge \ldots \wedge P(n)$ assuming UoD is $\{1 \ldots n\}$

Proposition $\forall x\ P(x)$ is **True** iff for every x in UoD, $P(x)$ is **True**

- **Existential Quantifier:** $\exists$
  - $\exists x\ P(x) \quad := \quad P(x)$ (is true) for some value(s) of x in the UoD
  - $\exists x\ P(x) :\simeq P(1) \vee P(2) \vee P(3) \vee \ldots \vee P(n)$ assuming UoD is $\{1 \ldots n\}$

Proposition $\exists x\ P(x)$ is **True** iff for at least one x in UoD, $P(x)$ is **True**