CS-210 Discrete Mathematics

Problem Set 9

1. Prove that any graph G is connected if and only if $\overline{G}$ is disconnected.

2. Show that any connected graph on n vertices has at least $n-1$ edges.

3. Let G be a simple graph. Show that an edge e is a cut edge, if and only if e is not part of any cycle in G.

4. Let D be a directed graph with strongly connected components $C_1, C_2, \ldots C_k$. Let $P = u, w_1, w_2, \ldots, w_s, v$ be a directed path from u to v, such that u and v belong to C_1. Show that for $1 \leq i \leq s$, w_i also belongs to C_1.

5. Let F be a forest with k components. Show that F has $n-k$ edges.

6. (a) Let G be a connected graph. Prove that in $G - v$ (the graph obtained by removing the vertex v and its incident edges), every connected component has a vertex w_i that is adjacent to v.

   (b) Let G be a connected graph with all its vertices have degree $\leq k$. Furthermore, there is a vertex in G which has degree $< k$. Prove that we can properly color G with k colors.

   (c) Let G be a connected graph such that all its vertices have degrees $\leq k$ (for $k \geq 0$) except one. That one vertex could have degree $> k$. Show that we can properly color G with $k+1$ colors.

7. Prove that a connected graph G with at least two vertices has an Euler circuit if and only every vertex of G has even degree.

8. Prove that a connected graph has an Euler path if and only if it has exactly two vertices of odd degree.

9. Let G be a connected bipartite planar graph with e edges and v vertices. Show that $e \leq 2v-4$.

10. Let G be a planar graph on v vertices and e edges. Suppose G is not connected and has two connected components each of size at least 3. Assume that a planar drawing of G divides the plane into f regions. Find a formula (similar to the Euler's formula for connected graph) for f in terms of v and e.

11. Prove the following theorem

    Theorem 1 *A full m-ary tree with*

    (a) n vertices has $i = (n-1)/m$ internal vertices and $l = [(m-1)n+1]/m$ leaves

    (b) i internal vertices has $n = mi+1$ vertices and $l = (m-1)i+1$ leaves

    (c) l leaves has $n = (ml-1)/(m-1)$ vertices and $i = (l-1)/(m-1)$ internal vertices.

12. We hinted at the proofs of all of the following statements about tree. In this question you are asked to give their formal but brief proofs. *Every tree $T = (V, E)$, $(|V| = n, |E| = m)$ satisfies the following properties*

(a) m=n-1

(b) There is a unique path between every pair of vertices

(c) T is an edge-maximal acyclic graph (i.e.Adding any edge to T creates a cycle)

(d) Every edge in T is a cut-edge

(e) T is an edge-minimal connected graph (i.e. Removing any edge disconnects T)

(f) T has at least two leaves.

13. Show that in every simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

14. Show that a simple graph with n vertices and k connected components has at most $\dfrac{(n-k)(n-k+1)}{2}$ edges.

15. *Fleury's algorithm* for constructing Euler circuit begins with an arbitrary vertex of a multigraph and forms a circuit by choosing edges successively. Once an edge is chosen, it is removed. Edges are chosen successively so that each edge begin where the last edge ends, and so that this edge is not a cut edge.

    Prove that Fluery's algorithm always produces an Euler circuit.

16. Show that if m and n are even positive integers, the crossing number of $K_{m,n}$ is less than or equal to $\dfrac{mn(m-2)(n-2)}{16}$

17. Suppose that a multigraph has $2m$ vertices of odd degree. Show that any circuit that contains every edge of the graph must contain at least m edges more than once.