CS-210 Discrete Mathematics

Problem Set 7

Unexplained answers will get no or little credit.

1. How many functions are there from the set $\{a_1, a_2, \ldots, a_n\}$, where n is a positive integer, to the set $\{0, 1\}$?

2. How many functions are there from the set $\{a_1, a_2, \ldots, a_n\}$ where n is a positive integer, to the set $\{b_1, b_2, \ldots, b_m\}$, where m is a positive integer?

3. How many **one-to-one** functions are there from the set $A = \{a_1, a_2, \ldots, a_n\}$ where n is a positive integer, to the set $B = \{b_1, b_2, \ldots, b_m\}$, where m is a positive integer?

4. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcome
   (a) are there in total?
   (b) contain exactly three heads?
   (c) contain at least three heads?
   (d) contain the same number of heads and tails?

5. How many bit strings of length 10 contain either five consecutive 0's or five consecutive 1's?

6. Prove the binomial theorem using mathematical induction.

7. Let $A = \{a_1, a_2, \ldots, a_{10}\}$ and $B = \{b_1, b_2, b_3\}$ be two sets.
   (a) How many different functions are there from A to B?
   (b) How many different functions are there from B to A?

8. How many triangles are determined by the vertices of a regular polygon of n sides? How many if no side of the polygon is to be a side of any triangle?

9. Provide a combinatorial proof for the following:
   For $n \geq 1$

$$2^n = \binom{n+1}{1} + \binom{n+1}{3} + \ldots + \begin{cases} \binom{n+1}{n}, & \text{n odd} \\ \binom{n+1}{n+1}, & \text{n even} \end{cases}$$

10. (a) How many nonnegative integer solutions are there to the pair of equations $x_1 + x_2 + x_+ \ldots + x_7 = 37$, $x_1 + x_2 + x_3 = 6$?
    (b) How many solutions in part (a) have $x_1, x_2, x_3 > 0$?

11. Let $p_1, p_2, p_3, \ldots, p_n \in \mathbb{Z}^+$. Prove that if $p_1 + p_2 + p_3 + ... + p_n - n + 1$ pigeons occupy n pigeonholes, then either the first pigeonhole has p_1 or more pigeons roosting in it, or the second pigeon has p_2 or more pigeons roosting in it,..., or the nth pigeonhole has p_n or more pigeons roosting in it.

12. Prove that at a party where there are at least two people, there are two people who know the same number of other people.

13. Let n and k be integers with $1 \leq k \leq n$. Show that $\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \frac{\binom{2n+2}{n+1}}{2} - \binom{2n}{n}$

14. Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$

15. **Theorem 1 :**
The number of different permutations on n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2,..., and n_k indistinguishable objects of type k, is
$$\frac{n!}{n_1!n_2!\dots n_k!}$$
Theorem 2 :
The number of the ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into boxes $i,\ i = 1, 2, \dots, k$, equals
$$\frac{n!}{n_1!n_2!\dots n_k!}$$
Prove Theorem 2 by first setting up a one-to-one correspondence between permutations of n objects with n_i indistinguishable objects of type i, $i = 1, 2, 3 \dots, k$ and the distributions of n objects in k boxes such that n_i objects are placed in boxes i, $i = 1, 2, 3, \dots, k$ and then applying Theorem 1.

16. How many ways are there for a horse race with four horses to finish if ties are possible? (*Note:* Any number of the four horses may tie)