CS-210 Discrete Mathematics

Problem Set 6

1. Prove by induction that for $n \geq 3$, any n-elements set has $n(n-1)(n-2)/6$ 3-elements subsets.

2. Suppose that five one and four zeros are arranged around a circle. Between any two equal bits you insert a 0 and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine zeros. [Hint: Work, backward, assuming that you did end up with nine zeros.]

3. Let $a_1, a_2, a_3, \ldots$ be (recursively) defined as follow:

   - Base cases: $a_1 = 3$ and $a_2 = 4$.
   - $a_i = a_{i-2} + 2a_{i-1}$

   Prove by strong induction that for every $k > 0$, a_{2k} is even and a_{2k-1} is odd.

4. What is wrong with this "proof"?

   "Theorem": Any set of $n \geq 2$ lines, no two of which are parallel to each other, intersect at a common point.

   "Basis Step": The statement is clearly true for $n = 2$, as if two lines are not parallel, then they intersect at exactly one (common) point.

   "Inductive Hypothesis": The statement is true for $n = k$; i.e. any set of k lines intersect at a common point.

   "Inductive Step": We will show that if any set of k non-parallel lines intersect at a common point, then any set of $k+1$ non-parallel lines intersect at a common point. Consider any set of $k+1$ lines, $l_1, l_2, l_3, \ldots, l_{k+1}$. By the inductive hypothesis, the (sub)set of lines $l_1, l_2, \ldots, l_k$ intersect at a common point, say that point is p_1. Also by the inductive hypothesis, the (sub)set of lines $l_2, l_3, \ldots, l_{k+1}$ intersect at a common point, say that point is p_2. We will show that $p_1 = p_2$. Indeed, if $p_1 \neq p_2$, then all lines containing both p_1 and p_2 must be the same (as two distinct points exactly determine a line, in other words if two lines have two distinct points in common, then those two lines are the same). Since all our lines are distinct, therefore we must have that $p_1 = p_2$. Hence all the $k+1$ lines contain the point $p_1 = p_2$, i.e. they intersect at the common point $p_1 = p_2$.

5. Prove that for every positive integer n

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} \; > \; 2(\sqrt{n+1} - 1)$$

6. Prove $2^n > n!$ for all $n \in \mathbb{Z}^+, n > 3$.

7. For $n \in \mathbb{Z}^+$, let H_n denote the nth harmonic number i.e $H_i = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{i}$

   (a) For all $n \in \mathbb{N}$ prove that $1 + (\frac{n}{2}) \leq H_{2^n}$

(b) Prove that for all $n \in Z^+$

$$\sum_{j=1}^{n} jH_j = \left[\frac{n(n+1)}{2}\right]H_{n+1} - \left[\frac{n(n+1)}{4}\right]$$

8. Let S_1 and S_2 be two sets where $|S_1| = m$, and $|S_2| = r$ $(m, r \in \mathbb{Z}^+)$ and the elements of S_1, S_2 are in ascending order. The elements in S_1 and S_2 can be merged into ascending order by making no more than $m+r-1$ comparisons. Use this result to establish the following.

   For $n \geq 0$, let S be a set with $|S| = 2^n$. Prove that the number of comparisons needed to place the elements of S is ascending order is bounded above by $n.2^n$.

9. Prove that for any positive integer n,

$$\sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{n+2}}{2^n} \text{ where } F_i \text{ is the ith Fibonacci number}$$

10. Let $L_0, L_1, L2, \ldots$ denote the Lucas numbers, where (1) $L_0 = 2, L_1 = 1$; and (2) $L_{n+2} = L_{n+1} + L_n$ for $n \geq 0$. When $n \geq 1$, prove that

$$L_1^2 + L_2^2 + L_3^2 + \ldots + L_n^2 = L_n L_{n+1} - 2$$

11. Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Let A and G be the arithmetic and geometric mean for these numbers respectively. Use mathematical induction to prove that $A \geq G$

12. Consider the sequence

$$d_n = \begin{cases} \dfrac{d_{n-1}^2}{d_{n-2}} & \text{if } n \geq 3 \\ 2 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \end{cases}$$

   Prove that $d_n = \dfrac{3^{n-1}}{2^{n-2}}$ for $n \geq 1$

13. Consider the sequence

$$r_n = \begin{cases} r_{n-1} + \dfrac{1}{r_{n-1}} & \text{if } n \geq 2 \\ 1 & \text{if } n = 1 \end{cases}$$

   Prove that $r_n \leq \sqrt{3n-2}$

14. For every positive integer there is a unique sequence of digits $d_0, d_1, \ldots, d_k$ that gives decimal representation of n.

15. Prove that

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq \sqrt{n}$$