CS-210 Discrete Mathematics

Problem Set 5

1. Give a proof by contradiction to the following statement.

   The sum of any rational number and any irrational number is an irrational number.

2. Give a proof by contrapositive to the following statement.

   For all positive integers n, if 5 does not divide n^2 then 5 does not divide n.

3. Let $r \geq 4$ be a positive integer. Prove that $\sqrt[r]{2}$ is irrational.

4. Prove that the sum of two positive integers of the same parity (odd/even) is even.
   Hint: Give a direct proof for each possible case.

5. Prove or disprove that, *if $a+b$ is an odd integer, then both $a+x$ and $b+x$ are odd integers*, where a, b and x are integers.

6. These definitions are needed for the following subproblems.

   - The **arithmetic mean** of two non-negative real numbers a and b is equal to $\frac{a+b}{2}$.
   - The **geometric mean** of two non-negative real numbers a and b is equal to $\sqrt{ab}$.
   - The **harmonic mean** of two real numbers a and b is equal to $\frac{2ab}{a+b}$.
   - The **quadratic mean** of two real numbers a and b is equal to $\sqrt{\frac{a^2+b^2}{2}}$.

   (a) By taking a few examples formulate a conjecture about the relative sizes of the arithmetic and geometric mean of a pair of real numbers (e.g. $AM(x, y)$ is always equal to $GM(x, y)$, or one is (strictly) less than the other etc.). Prove your conjecture.

   (b) Repeat part(a) for harmonic mean and geometric mean.

   (c) Repeat part(a) for quadratic mean and arithmetic mean.

7. Give a proof by contradiction to the following statement.

   There is no rational number r for which $r^3 + r + 1 = 0$.

   Hint: Assume that $r = \frac{a}{b}$ satisfies the above equation, and r is in lowest terms. Multiply the above equation with b^3 to get an equation involving integers, then consider the cases when a and b are odd/even.

8. Prove that these four statements about the integer n are equivalent:

   (a) n^2 is odd.

   (b) $1 - n$ is even.

   (c) n^3 is odd.

(d) $n^2 + 1$ is even.

Hint: Prove a series of if-then statement. First prove that if (a), then (b), second prove that if (b), then (c), and so on, and complete the cycle by proving if (d), then (a). Convince yourselves that this prove a bi-implication between the given statements (meaning they are equivalent)

9. Prove that if x and y are real numbers, then $max(x, y) + min(x, y) = x + y$.
   Hint: Use a proof by cases

10. Prove the **triangular inequality**, which states that if x and y are real numbers then $|x|+|y| \geq |x + y|$.

11. Prove the following statement by contradiction

$$\text{If } a, b \in \mathbb{Z}, \text{ then } a^2 - 4b \neq 2$$

12. Prove the following statement by contradiction

$$\text{For every real number } x \in [0, \pi/2] \text{ we have } sin(x) + cos(x) \geq 1$$

13. Give a direct proof for the following statement

Every nonzero rational number can be expressed as a product of two irrational numbers

14. Prove that the product of five consecutive integers is divisible by 120.

15. Prove by contraction that

$$\text{There exist no integers } a \text{ and } b \text{ for which } 21a + 30b = 1$$

16. Prove by contradiction that

$$\text{For every } n \in \mathbb{Z}, \text{ 4 divides } (n^2 + 2)$$