CS-210 Discrete Mathematics

Problem Set 3

1. For A, B, $C \subseteq \mathcal{U}$ Prove that
$$A \times (B - C) = (A \times B) - (A \times C)$$

2. Let A, B,C,D be non empty sets.
   (a) Prove that $A \times B \subseteq C \times D$ if and only if $A \subseteq C$ and $B \subseteq D$.
   (b) What happens to the results of (a) if any of the sets A, B,C,D is empty?

3. For each of the following functions, determine whether it is one-to-one and determine its range.
   (a) f: $\mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 1$
   (b) f: $\mathbb{Q} \to \mathbb{Q}$, $f(x) = 2x + 1$
   (c) f: $\mathbb{Z} \to \mathbb{Z}$, $f(x) = x^3 - x$
   (d) f: $\mathbb{R} \to \mathbb{R}$, $f(x) = e^x$
   (e) f:$[-\pi/2, \pi/2] \to \mathbb{R}$, $f(x) = sin(x)$
   (f) f: $[0, \pi] \to \mathbb{R}$, $f(x) = sin(x)$

4. Let $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ be two functions. Explain why the following are functions.
   - $h : \mathbb{Z} \to \mathbb{Z}$ defined as $h(x) = f(g(x))$ when $g(x)$ is an **onto** function.
   - $h : \mathbb{Z} \to \mathbb{Z}$ defined as $h(x) = f(x) + g(x)$.
   - $h : \mathbb{Z} \to \mathbb{Z}$ defined as $h(x) = f(x) \cdot g(x)$.

5. Prove that for a finite set X any **one-to-one** function $f : X \to X$ is **bijective**.

6. Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$. Suppose that $a > 0$ and $b > 0$. Give a counter example to show that the following are not valid equalities or show that they are valid.
   - $\lceil (a + b) \cdot (a - b) \rceil = \lceil (a + b) \rceil \cdot \lceil (a - b) \rceil$
   - $\lfloor \frac{(a-3)^2}{2} \rfloor = \lfloor \frac{a^2}{2} \rfloor - 3a + 5$
   - $\lfloor \frac{a-b}{2} \rfloor = \lfloor \frac{a}{2} \rfloor - \lfloor \frac{b}{2} \rfloor$

7. Show that for a function $f : X \to Y$, f^{-1} exists if and only if f is bijective.

8. For each of the following functions f: $\mathbb{R} \to \mathbb{R}$, determine whether f is invertible, and, if so, determine f^{-1}
   (a) $f = \{(x, y)|2x + 3y = 7\}$
   (b) $f = \{(x, y)|ax + by = c, b \neq 0\}$
   (c) $f = \{(x, y)|y = x^3\}$
   (d) $f = \{(x, y)|y = x^4 + x\}$

9. Prove that if $f : A \to B$, $g : B \to C$ are invertible functions, then $g \circ f : A \to C$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$