CS-210 Discrete Mathematics

Problem Set 2

1. For the universe of all integers, let $p(x), q(x), r(x), s(x)$ and $t(x)$ be the following statements:

$p(x)$:	x is even
$q(x)$:	x is (exactly) divisible by 4
$r(x)$:	x is (exactly) divisible by 5
$s(x)$:	$x > 0$
$t(x)$:	x is a perfect square

Write the following statements in symbolic form:

(a) At lease one integer is even.

(b) There exists a positive integer that is even.

(c) If x is even, then x is not divisible by 5.

(d) No even integer is divisible by 5.

(e) There exists an even integer divisible by 5.

(f) if x is even and x is a perfect square, then x is divisible by 4.

2. Consider the universe of all polygons with three or four sides, and define the following open statements for this universe.

$a(x)$:	all interior angles of x are equal
$e(x)$:	x is an equilateral triangle
$h(x)$:	all sides of x are equal
$i(x)$:	x is an isosceles triangle
$p(x)$:	x has an interior angle that exceeds 180
$q(x)$:	x is a quadrilateral
$r(x)$:	x is a rectangle
$s(x)$:	x is a square
$t(x)$:	x is a triangle

Translate each of the following statements into an English sentence, and determine whether the statement is true or false.

a) $\forall x\ [q(x) \oplus t(x)]$

b) $\forall x\ [i(x) \rightarrow e(x)]$

c) $\exists x\ [t(x) \wedge p(x)]$

d) $\forall x\ [(a(x) \wedge t(x)) \leftrightarrow e(x)]$

e) $\exists x\ [q(x) \wedge \neg r(x)]$

f) $\exists x\ [r(x) \wedge \neg s(x)]$

g) $\forall x\ [h(x) \rightarrow e(x)]$

h) $\forall x\ [t(x) \rightarrow \neg p(x)]$

i)$\forall x\ [s(x) \leftrightarrow (a(x) \wedge h(x))]$

j) $\forall x\ [t(x) \rightarrow (a(x) \leftrightarrow h(x))]$

3. Let $p(x, y)$, $q(x, y)$ and $r(x, y)$ represents three open statements, with the replacement for variables x, y chosen from some prescribed universe(s). What is the negation of the following statement?

$$\forall x \exists y \; [(p(x,y) \wedge q(x,y)) \rightarrow r(x,y)]$$

Show all the steps.

4. Let p(x,y), q(x,y) denote the following statements.

$$p(x,y): \quad x^2 \leq y \qquad\qquad q(x,y): \quad x+2 < y$$

In the universe for each of x,y consists of all real numbers, determine the truth value of each of the following statements.

a) $p(2,4)$

b) $q(1,\pi)$

c) $p(-3,8) \wedge q(1,3)$

d) $p(\frac{1}{2},\frac{1}{3}) \vee \neg q(-2,-3)$

e) $p(2,2) \rightarrow q(1,1)$

f) $p(1,2) \leftrightarrow \neg q(1,2)$

5. Let the universe of discourse for both variables be the set of integers. Determine whether the following universally quantified statements are **true** or **false**. If **false** give one counter-example for each part.

   (a) $\forall\, x\, \forall\, y\; (\neg(x = y) \rightarrow \neg(x^2 = y^2))$

   (b) $\forall\, x\, \exists\, y\; (x = y^3)$

   (c) $\forall\, x\, \exists\, y\; (1/x = y)$

   (d) $\forall\, x\, \exists\, y\; (y^3 < 100 + x)$

   (e) $\forall\, x\, \forall\, y\; (xy = y)$

   (f) $\forall\, x\, \forall\, y\; (x^3 \neq y^2)$

6. Let $G(x,y)$ be the predicate "x is a good friend of y". Let the UoD be all students in LUMS. Using quantifier negation prove that $\neg\exists\, x\, \forall\, y\; G(x,y) \equiv \forall\, x\, \exists\, y\; \neg G(x,y)$

7. Suppose a software system has 9 components $\{A, B, C, D, E, F, G, H, I\}$. Each component has either exactly one of the two types of bugs (Bug 1 and Bug 2) or has no bug (is clean). We want to identify which components have Bug 1 or Bug 2 or is clean. The software testing team has summarized its findings as follows.

- (7.1) Let $P(x)$ be the predicate that component x has Bug 1, let $Q(x)$ be the predicate that component x has Bug 2 and let $R(x)$ be the predicate that component x is clean. Translate each of the below findings in terms of the predicates $P(x)$, $Q(x)$ and $R(x)$.

   (a) E and H do not have the same bug.

   (b) If G has Bug 1 then all components have Bug 1.

   (c) If E has Bug 1 then H has Bug 1 too.

   (d) If C has Bug 1 then D and F do not have Bug 1.

(e) If either E or H has Bug 1 then I does not have Bug 2.

(f) At least 4 components have Bug 1.

(g) If A has either bug then all components have Bug 2.

(h) A and F are not in the same category.

(i) B has Bug 2.

(j) At least one of C and G have the same bug as B.

(k) Exactly 2 components have Bug 2.

(l) If I has bug 2 then at least one of D, F and A have Bug 2 too.

(m) If E or G have bug 2 then all components have either Bug 1 or Bug 2.

- (7.2) Determine which components have Bug 1, which ones have Bug 2 and which are clean.

  Hint: First determine which components have Bug 1 *then determine which ones have Bug* 2 *and then list the clean ones.*

8. (a) Prove the following DeMorgan's law for sets using membership table.

$$\overline{A \cup B \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$$

   (b) Let A, B and C be any sets. Prove the following identity using membership table.

$$\overline{(A \cap B) \cup (\overline{A} \cap C)} = (A \cap \overline{B}) \cup (\overline{A} \cap \overline{C})$$

   (c) Let A, B, and C be any sets. Prove the following identity by showing that each side is a subset of the other side.

$$(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C.$$

   (d) Let A, B and C be any sets. Show using set identities that

$$\overline{(A \cup C) \cap B} = \overline{B} \cup (\overline{C} \cap \overline{A}).$$

   (e) Let A, B and C be any sets. Show using set identities that

$$\overline{\overline{(A \cup B) \cap C} \cup \overline{B}} = B \cap C$$

9. Prove or disprove each of the following for set A, B $\subseteq \mathcal{U}$

   (a) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$
   (b) $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$

10. Write down all elements of the following two sets, A and B, where

$$A = \mathcal{P}(\mathcal{P}(\emptyset)),$$

$$B = \mathcal{P}(A),$$

and

$$C = \mathcal{P}(\mathcal{B}).$$