CS-210 Discrete Mathematics **Fall 2015**

1. What is the coefficient of x^2y^5 in $(6x-9y)^7$?

2. Prove the Binomial Theorem using mathemathical induction.

3. a)Let n be a positive integer, then prove/disprove:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$$

   b) Subsequently, prove that:

$$\sum_{k=0}^{n}2^k\binom{n}{k}=3^n$$

4. Prove or disprove the following: (hockey stick identity)

$$\sum_{k=0}^{r}\binom{n+k}{k}=\binom{n+r+1}{r}$$

   Whenever n and r are positive integers.
   a) prove with a combinatorial argument

   b) prove with Pascal's identity

5. Show that $\binom{n}{k}\leq 2^n$ for all positive integers n and all integers k with $0\leq k\leq n$.

6. Give a combinatorial proof that $\sum_{k=1}^{n}k\binom{n}{k}=n2^{n-1}$.
   (*Hint:* Count in two ways the number of ways to select a committee and to then select a leader of the committee.)

7. Give a combinatorial proof that $\sum\limits_{k=1}^{n}k\binom{n}{k}^2=n\binom{2n-1}{n}$.

8. a) How many permutations exist for the word "BANANAS" ?

   b) You write down all of these permutations, imposing a lexicographic (alphabetical) ordering. What would be the 244^{th} word you write?

9. What is the coefficient of $x^8\ y^9$ in the expansion of $(3x+2y)^{17}$

10. How many permutations of the 26 letters of English alphabet do not contain any of the strings $fish$, rat or $bird$.

11. How many elements are in the union of four sets if the sets have 50, 60, 70 and 80 elements, respectively, each pair of the sets has 5 elements in common, each triple of the sets has 1 common element, and no element is in all four sets?

12. Show that if seven integers are selected from the first ten positive integers, there must at least two pairs of these integers with a sum equal to 11.

    (a) Is the conclusion in the original statement true if six integers are selected rather than seven.

13. There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

14. A palindromic integer or palindrome is a positive integer whose decimal expansion is symmetric and that is not divisible by 10. In other words, it is an integer that reads the same backward as forward. For example, the following integers are all palindromes: 1,8,11,99,101,131,999,1234321
    a) How many five digit palindromes are there?
    b) How many are odd? How many are even?

15. How many strings of 10 ternary digits(0,1 or 2) are there that contain exactly two 0s, three 1s and five 9s.

16. Prove that:
    $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$

17. A certain family has 6 children, consisting of 3 boys and 3 girls. Assuming that all birth orders are equally likely, what is the number of ways that the 3 eldest children are the 3 girls?

18. You have n pennies to divide among k children, in how many ways can you distribute this money if

    (a) Each child must have at least one penny

    (b) The above constraint is not applied

19. Let n and k be integers with $1 \leq k \leq n$. Show that $\sum\limits_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \dfrac{\binom{2n+2}{n+1}}{2} - \binom{2n}{n}$

20. Give a combinatorial proof that $\sum\limits_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n}$

21. Let $p_1, p_2, p_3, \ldots, p_n \in \mathbb{Z}^+$. Prove that if $p_1 + p_2 + p_3 + ... + p_n - n + 1$ pigeons occupy n pigeonholes, then either the first pigeonhole has p_1 or more pigeons roosting in it, or the second pigeon has p_2 or more pigeons roosting in it,..., or the nth pigeonhole has p_n or more pigeons roosting in it.

22. How many bit strings of length 8 do not contain 6 consecutive $0s$?

23. How many functions are there from the set $\{a_1, a_2, \ldots, a_n\}$, where n is a positive integer, to the set $\{0, 1\}$?

24. There are four transport companies that operate between Islamabad and Lahore, and three transport companies between Lahore and Karachi. Find the number m of ways that a man can travel by bus:

    (a) from Islamabad to Karachi by way of Lahore

    (b) round trip from Islamabad to Karachi by way of Lahore

    (c) round trip from Islamabad to Karachi by way of Lahore but without using a transport company more than once.

25. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes

    (a) are there in total?

    (b) contain exactly three heads?

    (c) contain at least three heads?

    (d) contain the same number of heads and tails?

26. Prove the binomial theorem using mathematical induction.

27. Let $A = \{a_1, a_2, \ldots, a_{10}\}$ and $B = \{b_1, b_2, b_3\}$ be two sets.

    (a) How many different functions are there from A to B?
    (b) How many different functions are there from B to A?

28. How many triangles are determined by the vertices of a regular polygon of n sides? How many if no side of the polygon is to be a side of any triangle?

29. Prove that at a party where there are at least two people, there are two people who know the same number of other people.

30. How many ways are there for a horse race with four horses to finish if ties are possible? (*Note:* Any number of the four horses may tie)

31. Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.

    (a) Is the conclusion in the original statement true if four integers are selected rather than five.

32. How many permutations of the 10 digits either begin with the 3 digits 987, contain the digits 45 in the fifth and sixth positions, or end with the 3 digits 123.

33. What is the coefficient of $x^{101}\ y^{99}$ in the expansion of $(2x - 3y)^{200}$

34. How many functions are there from the set $\{a_1, a_2, \ldots, a_n\}$ where n is a positive integer, to the set $\{b_1, b_2, \ldots, b_m\}$, where m is a positive integer?

35. How many bit strings of length 10 contain either five consecutive 0's or five consecutive 1's?

36. Provide a combinatorial proof for the following:
    For $n \geq 1$

$$2^n = \binom{n+1}{1} + \binom{n+1}{3} + \ldots + \begin{cases} \binom{n+1}{n}, & \text{n odd} \\ \binom{n+1}{n+1}, & \text{n even} \end{cases}$$

37. (a) How many nonnegative integer solutions are there to the pair of equations $x_1 + x_2 + x_+ \ldots + x_7 = 37$, $x_1 + x_2 + x_3 = 6$?

    (b) How many solutions in part (a) have $x_1, x_2, x_3 > 0$?

38. **Theorem 1 :**
    The number of different permutations on n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2,..., and n_k indistinguishable objects of type k, is

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Theorem 2 :
The number of the ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are placed into boxes i, $i = 1, 2, \ldots, k$, equals

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Prove Theorem 2 by first setting up a one-to-one correspondence between permutations of n objects with n_i indistinguishable objects of type i, $i = 1, 2, 3 \ldots, k$ and the distributions of n objects in k boxes such that n_i objects are placed in boxes i, $i = 1, 2, 3, \ldots, k$ and then applying Theorem 1.

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39. Suppose that five *ones* and four *zeros* are arranged around a circle. Between any two equal bits you insert a 0 and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine *zeros*. [Hint: Work backward, assuming that you did end up with nine *zeros*.]

40. A *palindrome* is a string whose reversal is identical to the string. How many bit strings of length n are palindromes?

41. The LUMS male hostel has 40 students - 12 are freshmen, 8 sophomores, and 20 juniors. How many ways can you put all 40 in a row for a picture, with all 12 freshmen on the left, all 8 sophomores in the middle, and all 20 juniors on the right.

42. How many bit strings (strings of 0s and 1s) are there of length 7 that have more 0s than 1s?

43. How many different passwords are possible if each password consists of six characters where each character is either an uppercase letter, a lowercase letter, or a digit, and at least one digit must be included in the password?

44. How many permutations of the letters $ABCDEFGH$ contain:

    (a) the string CDE?

    (b) the strings BA and FGH?

    (c) at least one of the strings BA or FGH?

    (d) the strings FGH and DG?

45. (a) How many ways are there to arrange n 1s and k 0s into a sequence?

    (b) How many solutions does $x_0 + x_1 + ... + x_k = n$ have, if all xs must be $non - negative$ integers?

    (c) How many solutions does $x_0 + x_1 = n$ have, if all xs must be *strictly positive* integers?

    (d) How many solutions does $x_0 + x_1 + ... + x_k = n$ have, if all xs must be *strictly positive* integers?

46. Prove that if n and k are integers with $1 \leq k \leq n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$, using:

    (a) using a combinatorial proof

(b) using an algebraic proof

47. Prove using a combinatorial argument

    (a) $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$

    (b) $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$

48. Prove that at a graduation party with at least two people, there are two that know the same number of party attendees.