Lahore University of Management Sciences

EE 203: Engineering Models

Fall 2023

Reading Assignment # 1: Introduction to Engineering Models with Differential Equations
The first two lectures (Lecture # 1) introduced the field of Engineering Models followed by few real-life applications involving differential equations (DEQs). This reading assignment will direct students to relevant chapter(s) and/or section(s) in the book(s) explaining in detail systems and experiments, model concept, simulation, types of models, building models, and the need of sustainable solutions. We elaborately discussed five examples, drawn from various fields, which ultimately boil down to a mathematical model involving DEQs. Students are most welcome to discuss offline with the instructor and the co-instructors at the assigned time(s)-and-date(s) mentioned in the Syllabus. The book(s), Syllabus, and PDF slides for Lectures are uploaded on LMS under the Resources tab.

Topic	**Book Section**
An Applied Invitation to Engineering Models	HMD, **Video (37:42 minutes)**
Need of Engineering Models	IMS, **Chapter 1**
Systems and Experiments	IMS, **Chapter 1.1**
The Model Concept	IMS, **Chapter 1.2**
Simulation	IMS, **Chapter 1.3**
Building Models	IMS, **Chapter 1.4**
Differential Equations: Definitions and Terminology	DE, **Chapter 1.2**
Differential Equations as Mathematical Models	DE, **Chapter 1.3**

References

Some books, notes, slides, videos, and research papers

- H. Mohy-ud-Din, **(HMD)**, **An Applied Invitation to Engineering Models**, EE 203: Engineering Models, LUMS, (2021). [Weblink]
- G. Zill & M. R. Cullen, **(DE)**, **Differential Equations with Boundary-Value Problems**, 7th edition, Cengage Learning, (2008). Chapter 1 – 1.4
- P. Fritzson **(IMS)**, **Introduction to Modeling and Simulation of Technical and Physical Systems with Modelica**, IEEE Press and John Wiley, (2011). Chapter 1.3

Prepared and Revised by:	**Dr. Hassan Mohy-ud-Din**
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Reading Assignment # 2: Differential Equations and Initial Value Problems

In Week 2, at first, we introduced the nomenclature for differential equations (DEQ) which will be frequently used in the rest of course. Everyone is expected to master this notation as it defines the language of the differential equations. Secondly, we studied classification of differential equations by type, order, and linearity with detailed examples. This included compact definitions of first-order ODE, $M(x,y)dx + N(x,y)dy = 0$, n^{th} order ODE, $F\left(x, y, y', y'', \cdots, y^{(n)}\right) = 0$, and the corresponding normal form, $\frac{d^n y}{dx^n} = f(x, y, y', y'', \cdots, y^{(n-1)})$. Likewise, everyone is expected to master these compact definitions.

Thirdly, we talked about the solution of an ODE. A solution of an ODE is $y = \phi(x)$ satisfying certain conditions (Lecture 3, Slide 17). Every solution is defined on an interval which is called the domain of the solution (amongst other names). We classified solution(s) to a DEQ as explicit solution(s) and implicit solution(s). Based on these classifications, we further studied trivial solution, particular solution, and singular solution. For an n^{th} order ODE we also studied an n-parameter family of solutions. Likewise, everyone is expected to master these concepts.

Fourthly, we studied in great detail the difference between the domain of the function and the domain of the solution of an ODE. This is extremely important as every solution is defined on a domain (aka interval). These concepts were studied numerically with the help of fifteen (15) examples.

Fifthly, we coupled a set of (n) initial conditions (constraints) with an n^{th} order ODE to form an n^{th} order Initial Value Problem (IVP). We studied solutions curves associated with an n^{th} order ODE and an n^{th} order IVP. We also defined the solution of an n^{th} order IVP. This solution is also defined on a domain (or interval) which also contains the initial point x_0 at which n initial conditions are also satisfied.

Sixthly, we again studied in great detail the difference between the domain of the function, the domain of the solution(s) of an ODE, and the domain of the solution of an IVP. Likewise, everyone is expected to master these concepts.

Lastly, we laid down conditions for the existence and uniqueness of the solution to a DEQ and IVP. We also elaborated on a theorem of existence of a unique solution. Concepts related to IVP were studied numerically with the help of five (5) examples.

With this we conclude our study of Chapter 1 of **DE**.

Students are most welcome to discuss offline with the instructor and the Teaching Assistant at the assigned time(s)-and-date(s) mentioned in the Syllabus. The book(s), Syllabus, and PDF slides for Lectures 3 and 4 are uploaded on LMS under the Resources tab.

Topic	**Book Section**
Differential Equations: Definitions and Terminology	DE, **Chapter 1.1**
Differential Equations: Initial Value Problems	DE, **Chapter 1.2**

References

Some Books, notes, slides, and research papers

- G. Zill & M. R. Cullen, **(DE)**, **Differential Equations with Boundary-Value Problems**, 7^{th} edition, Cengage Learning, (2008). Chapter 1

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Reading Assignment # 3: Direction Fields, Autonomous DEQs, and Separable Variables

In Week 3, we started with the discussion on Direction Fields. Direction fields is a truly imaginative approach of gaining a qualitative understanding of the properties of the solutions to DEQ without even attempting to solve the DEQ algebraically. Direction field gives the appearance and shape of the family of solution curves of a DEQ. It displays region(s) of unusual behavior. The solution curve follows the flow pattern of the direction field.

To plot/sketch the direction field, we explore a novel use of the slop function which is the RHS of a first-order differential equation. Using this slope function, we can compute (local) gradient/slope/inclination at any point on a 2-D Cartesian grid. Arrows are used to show if the function is increasing, decreasing, or constant.

Later, we introduced a new class of DEQ, called Autonomous DEQ, where the slope function is only a function of the dependent variable i.e. not guided by the independent variable. We concluded that the roots of the slope function (in first-order ODE) are the critical points or equilibrium solutions of the autonomous DEQ. In the case of autonomous DEQ, one can plot a phase portrait which shows regions of increasing and decreasing function value. Critical points mark points with zero rate of change.

Direction fields and phase portraits are novel ways of depicting the shape of the family of solutions for DEQ.

We concluded Week 3, by discussing the first algebraic approach of solving DEQ and IVP – Separable Variables. In the case of first-order DEQ, if the slope function can be factored as a product of univariate functions (depending on one variable only), one could use the method of integration for computing the solutions of the DEQ and IVP. We also studied an important aspect of loosing a solution which happens in the case of autonomous DEQ. One can obtain the lost solution by computing the critical point(s) of the slope function of an autonomous DEQ. We capped the lecture by solving examples where the antiderivative cannot be obtained in a closed analytical form.

Students are most welcome to discuss offline with the instructor and the Teaching Assistant at the assigned time(s)-and-date(s) mentioned in the Syllabus. The book(s), Syllabus, and PDF slides for Lectures 5 and 6 are uploaded on LMS under the Resources tab.

Topic	Book Section
Differential Equations: Direction Fields	DE, **Chapter 2.1**
Differential Equations: Separable Variables	DE, **Chapter 2.2**

References

Some Books, notes, slides, and research papers

- G. Zill & M. R. Cullen, **(DE)**, **Differential Equations with Boundary-Value Problems**, 7th edition, Cengage Learning, (2008). Chapter 2

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Reading Assignment # 4: Linear Equations, Exact Equations, and Solutions by Substitutions

In Week 4, we solved linear first-order differential equations using Variation of Parameters method. This method proceeds as follows: (1) Re-write the first-order DEQ in standard form, $\frac{dy}{dx} + P(x)y = f(x)$. (2) Compute the integration factor $e^{\int P(x)dx}$. (3) Multiple the DEQ in standard form with the integration factor to reach compact representation, $\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x)$. (4) By integrating on both sides, we obtain the solution to a linear first-order DEQ. This method yields a complete solution (aka general solution) consisting of the complementary solution and the particular solution, $y(x) = \underbrace{ce^{-\int P(x)dx}}_{y_c} + \underbrace{e^{-\int P(x)dx}\int e^{\int P(x)dx}f(x)\,dx}_{y_p}$. This complete solution belongs to a one-parameter family of explicit solutions. One can use the initial condition from an IVP to solve for the constant c. If $f(x) = 0$, it is a homogeneous (linear) first-order DEQ, otherwise, it is a nonhomogeneous (linear) first-order DEQ.

We also looked at autonomous linear first-order DEQ where critical points must be computed as they form the equilibrium solutions of the underlying DEQ.

We followed it up by studying the existence and uniqueness of the solution of IVP defined by linear first-order DEQs. We concluded that if $P(x)$ and $f(x)$ are continuous than the solution exists and is unique. This implies that each initial point (condition) has a unique c.

While using the Variation of Parameters method, one must always be careful while computing the integration factor. The integration factor may only be defined on an interval which ultimately defines the interval where the solution is define. **This is extremely important**.

In Week 4 – 5, we studied a specific approach of solving exact first-order DEQ. In this approach, we focus on the differential form of the first-order DEQ, $M(x,y)dx + N(x,y)dy = 0$. If this is an exact equation than the LHS can be compact represented as a differential of a function of two variables i.e., $d(f(x,y)) = M(x,y)dx + N(x,y)dy = 0$. If this is the case one can simply perform integration twice (once with respect to x and another with respect to y) to obtain a one-parameter family of implicit solutions.

By this time, we have learned three approaches for solving first-order DEQs: (1) separability, (2) linearity, and (3) exactness. This begs the following question: What if none of the three features are present?

We capped Week 5 by learning a new approach called Solutions by Substitutions. The gist of this approach is to transform the original first-order DEQ (by substitution) into a separable first-order DEQ OR linear first-order DEQ OR exact first-order DEQ. A key tool used in this method is homogeneity of functions. A function is homogeneous of degree $\alpha \in \mathbb{R}$ if $f(tx, ty) = t^{\alpha} f(x,y)$.

NOTE: Homogeneity of functions must not be confused with homogeneity of DEQ defined above.

If $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree, one can substitute $y = ux$ to obtain a separable first-order DEQ in u. After solving for u, one can resubstitute to obtain the solution $y(x)$.

If the first-order DEQ can be written in a form of Bernoulli's equation, $\frac{dy}{dx} + P(x)y = f(x)y^n$, then substituting $u = y^{1-n}$ reduces any Bernoulli's equation to a linear equation. In this scenario, one can resort to the method of Variation of Parameters to solve linear first-order DEQs.

Lastly, if the first-order DEQ can be written as $\frac{dy}{dx} = f(Ax + By + C)$, then substituting $u = Ax + By + C$ for $B \neq 0$ always yields a separable first-order DEQ.

This concludes Chapter # 2 (**DE**).

Students are most welcome to discuss offline with the instructor and the Teaching Assistant at the assigned time(s)-and-date(s) mentioned in the Syllabus. The book(s), Syllabus, and PDF slides for Lectures 7, 8, and 9 are uploaded on LMS under the Resources tab.

Topic	Book Section
Differential Equations: Linear Equations	DE, **Chapter 2.3**
Differential Equations: Exact Equations	DE, **Chapter 2.4**
Differential Equations: Solutions by Substitutions	DE, **Chapter 2.5**

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References
Some Books, notes, slides, and research papers
• G. Zill & M. R. Cullen, **(DE)**, **Differential Equations with Boundary-Value Problems**, 7^{th} edition, Cengage Learning, (2008). Chapter 2

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Reading Assignment # 5: Modeling with First-order Differential Equations

We moved to a core component of the course i.e., Engineering Modeling with DEQs. We looked at the following real applications involving first-order DEQs: (1) population growth (and decline), (2) capital growth (and decrease), (3) radioactive decay, (4) bacterial growth, (5) Newton's law of cooling (or warming), (6) mixture of solutions, (7) population dynamics using logistic equation, (8) chemical reactions, and (9) predator-prey model. We learned to formulate DEQ modeling the afore-mentioned real-life phenomenon and solved them using the methods/algorithms/approaches covered in the course.

Students are most welcome to discuss offline with the instructor and the Teaching Assistant at the assigned time(s)-and-date(s) mentioned in the Syllabus. The book(s), Syllabus, and PDF slides for Lecture 10 are uploaded on LMS under the Resources tab.

Topic	**Book Section**
Linear Models	DE, **Chapter 3.1**
Non-Linear Models	DE, **Chapter 3.2**

References

Some Books, notes, slides, and research papers

- G. Zill & M. R. Cullen, **(DE)**, **Differential Equations with Boundary-Value Problems**, 7th edition, Cengage Learning, (2008). Chapter 3

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Reading Assignment # 6: Higher-order Differential Equations

Let us remind ourselves of two overarching goals of this course. We would like to construct mathematical models of various real-life phenomena in the form of DEQs **and** be able to solve them. We explored few real-life examples in Lectures 2 and 3 and derived the corresponding DEQs. We also saw few questions in the midterm exam (Problems 6 and 7) that extended some of these models by adding more naturally occurring factors. We will surely comeback to discuss more engineering models in detail in Chapter 3.

As far as solution(s) of DEQ is concerned our goal is to be able to compute general solutions (complementary + particular) of nonhomogeneous DEQs. We have, so far, completed our study of first-order linear first-order DEQs. We can use the method of Variation of Parameters to compute the complementary and particular solutions of nonhomogeneous linear, first-order DEQs (Chapter 2). We have also completed our study of nonhomogeneous constant-coefficient, linear, higher-order DEQs. We can compute the complementary solution using the corresponding auxiliary equation and the particular solution by the method of undetermined coefficients (Chapter 4).

This week we completed our study of nonhomogeneous constant-coefficient, linear, second-order DEQs. We can compute the complementary solution using the corresponding auxiliary equation and the particular solution by the method of Variation of Parameters (Chapter 4).

We capped last week by studying a new class of DEQs called Cauchy-Euler DEQ. Here, we explore solutions of the form $y(x) = x^m$. For Cauchy-Euler DEQs (of any order) we can compute the complementary solution using the corresponding auxiliary equation. **An important thing to note here is that the auxiliary equation for homogeneous Cauchy-Euler DEQ are quite different and cannot be read of easily**. However, so far, we can only compute particular solution for second-order Cauchy-Euler DEQ using the method of Variation of Parameters (Chapter 4).

We will cover section 4.8 this week which will conclude Chapter # 4 (**DE)**.

Students are most welcome to discuss offline with the instructor and the Teaching Assistant at the assigned time(s)-and-date(s) mentioned in the Syllabus. The book(s), Syllabus, and PDF slides for Lectures 11 – 18 are uploaded on LMS under the Resources tab.

Topic	**Book Section**
Preliminary Theory – Linear Equations	DE, **Chapter 4.1**
Reduction of Order	DE, **Chapter 4.2**
Homogeneous Linear Equations with Constant Coefficients	DE, **Chapter 4.3**
Undetermined Coefficients – Superposition Approach	DE, **Chapter 4.4**
Variation of Parameters	DE, **Chapter 4.6**
Cauchy – Euler Equation	DE, **Chapter 4.7**

References

Some Books, notes, slides, and research papers

- G. Zill & M. R. Cullen, **(DE)**, **Differential Equations with Boundary-Value Problems**, 7th edition, Cengage Learning, (2008). Chapter 4

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Lahore University of Management Sciences

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Reading Assignment # 7: Modeling with First-order and Higher-order Differential Equations
After finishing Chapter # 4 **(DE)** we moved to a core component of the course i.e., Engineering Modeling with DEQs. Firstly, we looked at the following real applications involving first-order DEQs: (1) population growth (and decline), (2) capital growth (and decrease), (3) radioactive decay, (4) bacterial growth, (5) Newton's law of cooling (or warming), (6) mixture of solutions, (7) population dynamics using logistic equation, (8) chemical reactions, and (9) predator-prey model. Secondly, we looked at the following real applications involving second-order DEQs: (1) spring-mass systems (free undamped motion, free damped motion, critically damped motion, driven motion, and variable spring constants), and (2) series circuit analogue – LRC. We learned to formulate DEQ modeling the afore-mentioned real-life phenomenon and also solved them using the methods/algorithms/approaches covered in the course. Students are most welcome to discuss offline with the instructor and the Teaching Assistant at the assigned time(s)-and-date(s) mentioned in the Syllabus. The book(s), Syllabus, and PDF slides for Lectures 10 and 19 are uploaded on LMS under the Resources tab.

Topic	**Book Section**
Linear Models	DE, **Chapter 3.1**
Non-Linear Models	DE, **Chapter 3.2**
Linear Models: Initial-Value Problems	DE, **Chapter 5.1**

References

Some Books, notes, slides, and research papers

- G. Zill & M. R. Cullen, **(DE)**, **Differential Equations with Boundary-Value Problems**, 7th edition, Cengage Learning, (2008). Chapters 3 and 5

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Reading Assignment # 8: Boundary-Value Problems in Rectangular Coordinates i.e. Partial Differential Equations

After a (hopefully) fascinating study (😊) of first-order and higher-order ODEs we moved to another stimulating journey (😊😊) of partial differential equations (PDEs). We restricted ourselves to linear second-order PDEs which appear in several real-life problems e.g. heat transfer as a function of position and time ($u(x,t)$), displacement in transverse vibrations as a function of position and time ($u(x,t)$), and steady-state distribution in 2D ($u(x,y)$).

Solutions of such PDEs are bi-variate functions depending on two independent variables. To solve such PDEs we appealed to separation of variables which factors the bi-variate function into a product of univariate functions. This reduces a linear second-order PDE into two separate ODEs which can be solved by a variety of methods covered in the course.

We classified linear second-order PDEs as hyperbolic, parabolic, and elliptic PDEs based on the coefficients of the second-order derivatives.

We explored in detail three revolutionary PDEs appearing in several real applications: (1) Heat equation, (2) Wave equation, and (3) Laplace equation. We derived these PDEs from the underlying principles of physics and associated boundary conditions and initial conditions to formulate the dynamics as a boundary-value problem (BVP). We paid special heed to several boundary conditions modeling physical constraints of the system: (1) Dirichlet boundary condition, (2) Neumann boundary condition, and (3) Robin boundary condition.

We proceeded to solve the formulated Heat BVP, Wave BVP, and Laplace BVP using separation of variables. Several examples were studied involving formulation of BVPs and obtaining their particular solutions. In the process we came across another revolutionary mathematical tool, called the Fourier series/expansion, which allows us to represent an arbitrary function (with finite amount of discontinuities and other regularity constraints beyond the scope of this course) as an infinite series sum of sines and cosines. Fourier series appears in numerous and numerous engineering applications and will stay with you for several semester, courses, and projects (😊).

Students are most welcome to discuss offline with the instructor and the Teaching Assistant at the assigned time(s)-and-date(s) mentioned in the Syllabus. The book(s), Syllabus, and PDF slides for Lectures 21 – 25 are uploaded on LMS under the Resources tab.

Topic	**Book Section**
Separable Partial Differential Equations	DE, **Chapter 12.1**
Classical PDEs and Boundary-value Problems	DE, **Chapter 12.2**
Heat Equation	DE, **Chapter 12.3**
Wave Equation	DE, **Chapter 12.4**
Laplace Equation	DE, **Chapter 12.5**

References

Some Books, notes, slides, and research papers

- G. Zill & M. R. Cullen, **(DE)**, **Differential Equations with Boundary-Value Problems**, 7th edition, Cengage Learning, (2008). Chapter 12

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