Quantitative Finance

Lecture 12

Black Scholes Formula

From Random Walks to the Heat Equation

- Consider a random walk
- Suppose at time t one is at position x
- Consider the probability distribution at the next time step

$$p(x,t+\Delta t) = \frac{1}{2}p(x+\Delta x,t) + \frac{1}{2}p(x-\Delta x,t)$$

Who ordered that……

- Taylor expanding etc…

$$= \frac{1}{2}\left\{p(x,t) + \frac{\partial p(x,t)}{\partial x}\Delta x + \frac{\partial^2 p(x,t)}{\partial x^2}\Delta x^2 + \cdots\right\}$$

$$+ \quad \left\{\frac{1}{2}p(x,t) - \frac{\partial p(x,t)}{\partial x}\Delta x + \frac{\partial^2 p(x,t)}{\partial x^2}\Delta x^2 + \cdots\right\}$$

$$\frac{p(x,t+\Delta t) - p(x,t)}{\Delta t} = \frac{\partial^2 p(x,t)}{\partial x^2}\frac{\Delta x^2}{\Delta t} + \frac{O(\Delta x^4)}{\Delta t}$$

- Taking vanishingly small time steps and noting we scaled so that $\frac{\Delta x^2}{\Delta t}$ remained finite

$$\frac{\partial p(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 p(x,t)}{\partial x^2}$$

The ‘Scaling’

- We note that to get the heat equation we needed the scaling $\frac{x}{\sqrt{t}}$
- This points towards the fact that in the heat equation the two variables appear in this particular combination
- We will exploit this in order to reduce the heat equation to an ordinary differential equation

Similarity Solution

- Consider the IVP

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty \qquad 0 \le \tau \le \frac{\sigma^2}{2} T$$

$$u(x,0) = \delta(x - x_0)$$

- Consider the similarity transform

$$u(x,t) = t^{-1/2} \phi(\xi)$$

  - With $\xi = \dfrac{x}{\sqrt{t}}$

Similarity Solution

- We note that

$$u_t(x,t) = -\frac{1}{2}t^{-3/2}\phi(\xi) - \frac{1}{2}t^{-3/2}\xi\phi'(\xi)$$

$$u_x(x,t) = t^{-1}\phi'(\xi)$$

$$u_{xx}(x,t) = t^{-3/2}\phi''(\xi)$$

- So that the IVP becomes

$$-\frac{1}{2}t^{-3/2}\phi(\xi) - \frac{1}{2}t^{-3/2}\xi\phi'(\xi) = t^{-3/2}\phi''(\xi)$$

Similarity Solution

- Simplifying we get

$$\xi\phi(\xi) = -\phi'(\xi)$$

- Which implies that

$$\phi(\xi) = \phi(0)e^{-\xi^2/2}$$

- Choose $\phi(0)$ such that $\int_{-\infty}^{\infty} \phi(\xi)d\xi = 1$

- Giving $\phi(0) = \dfrac{1}{\sqrt{2\pi}}$

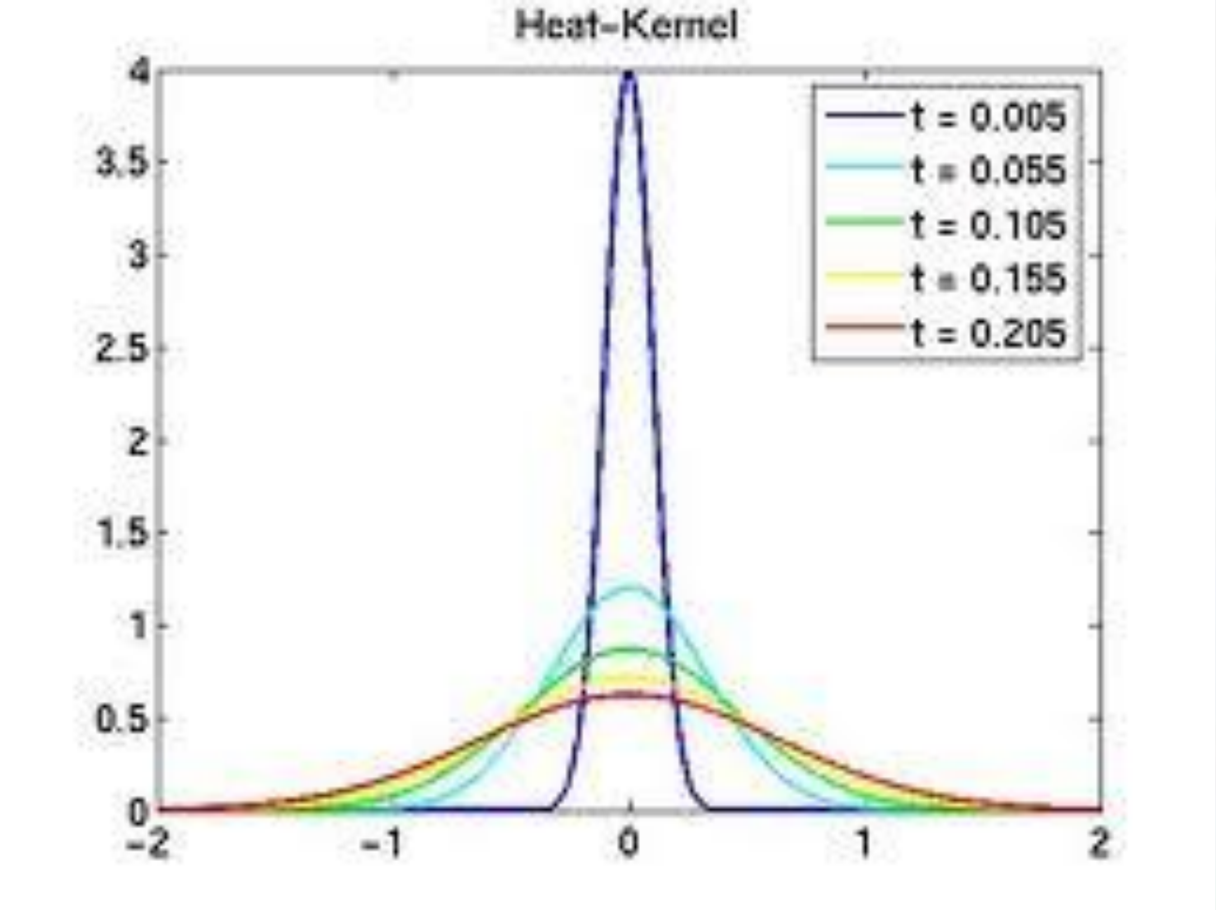
Similarity Solution

- In terms of the original variables

$$u(x,t) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}$$

- This is a ‘delta sequence’ introduced in the previous lecture
- This satisfies the delta function initial condition!!

The Heat Kernel



Solving the Heat Equation

- For a general initial condition $u(x,0) = f(x)$ we note that the solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-x_0)^2}{2t}} f(x_0)\,dx_0$$

- The idea is that you ‘break’ your IC into tiny bits and add then (integrate)
- See “A small note on Green’s Functions” on the course webpage for more details

Back to the Black Scholes Equation

- We had transformed the B-S FVP to the IVP

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty \qquad 0 \le \tau \le \frac{\sigma^2}{2} T$$

$$u(x,0) = e^{-\alpha x} f(e^x)$$

- Where

$$V(S,t) = v\left(\ln(S), \frac{\sigma^2}{2}(T-t)\right)$$

$$= e^{-\alpha \ln(S) - \beta \frac{\sigma^2}{2}(T-t)} u\left(\ln(S), \frac{\sigma^2}{2}(T-t)\right)$$

Solving the Black Scholes for a European Call

- Here the final condition is replaced by the IC

$$u(x,0) = e^{-\alpha x} f(e^x) \; = e^{-\alpha x} \max\left(e^x - K, 0\right)$$

$$= \max\left(e^{(1-\alpha)x} - K e^{-\alpha x}, 0\right)$$

- Recalling the IVP and the fundamental solution, we have

$$u(x,\tau) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{(x-x_0)^2}{2\tau}} f(e^{x_0}) dx_0$$

Solving the Black Scholes for a European Call

- Going back to the original variables

$$V(S,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_0^\infty e^{\frac{-\left(\log\left(S/S_0\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}} f(S_0) \frac{dSx_0}{S_0}$$

- The call option has the payoff

$$f(S) = \max(S-K, 0)$$

Solving the Black Scholes for a European Call

- Substituting into the solution we have

$$V(S,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_K^\infty e^{\frac{-\left(\log\left(S/S_0\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}} (S_0 - K)\frac{dSx_0}{S_0}$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_{\log(K)}^\infty e^{\frac{-\left(-x'+\log(S)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}} (e^{x'} - K)dx'$$

Solving the Black Scholes for a European Call

- From which we get

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_{\log(K)}^{\infty} e^{\frac{-\left(-x'+\log(S)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}} e^{x'}\,dx'$$

$$- K\frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_{\log(K)}^{\infty} e^{\frac{-\left(-x'+\log(S)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}} dx'$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_{\log(K)}^{\infty} e^{\frac{-\left(-x'+\log(S)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}} e^{x'}\,dx'$$

$$- K\frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_{\log(K)}^{\infty} e^{\frac{-\left(-x'+\log(S)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}} dx'$$

Solving the Black Scholes for a European Call

- These are integrals of the form $\int_d^\infty e^{\frac{-x'^2}{2}}\,dx'$

- By doing a little algebra (HW 2) we have

$$V(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$d_2 = \frac{\log(S/K) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_1 = \frac{\log(S/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

European Call Option