Riemann sums and definite integrals

S Sial

Dept of Mathematics
LUMS

Fall 2020-2021

Outline

Definite integrals

Definite integrals

Definite integrals

Area under the graph of
$f : [a, b] \rightarrow \mathbb{R}$

Areas below the axis are taken to be negative.

Definite integrals

Area under a curve contd

$$\int_{x=a}^{x=b} f(x)\,dx$$

Definite integrals

Example

A previous integral that we evaluated was

$$\int_{x=0}^{x=1} x\,dx \quad = \quad \frac{1}{2}$$

Riemann sums

Riemann sums

Riemann sums

Evaluate

$$\int_{x=a}^{x=b} f(x)\,dx$$

if the definite integral exists.

Riemann sums

Partition

$$P = \{x_0 = a, x_1, x_2, \cdots x_n = b\}$$

x_k divide $[a, b]$ into n subintervals.

Riemann sums

Partition contd

$$||P|| \;=\; \max\{x_{i+1} - x_i\}$$

Riemann sums

Width of the i th rectangle

$$\Delta_i \;=\; x_i - x_{i-1}$$

Riemann sums

Riemann sum

$$S = \sum_{i=1}^{n} f(\zeta_i)(x_i - x_{i-1})$$

where the ζ_i are taken in the intervals $[x_i, x_{i-1}]$.

Riemann sums

Definite integral

$$\int_{x=a}^{x=b} f(x)\,dx \;=\; \lim_{||P||\to 0} \sum_{i=1}^{n} f(\zeta_i)(x_i - x_{i-1})$$

One method to evaluate

The area

$$\int_{x=a}^{x=b} f(x)\,dx$$

n rectangles of equal width

n rectangles of equal width

Width

$$\Delta = \frac{b-a}{n}$$

Partition

$P =$

$$\{x_0 = a, x_1, x_2, \ldots, x_{n-1}, x_n\}$$

With

$$x_k = a + k\Delta$$

Height of the kth rectangle

Evaluations at

$$\zeta_k = x_k = a + k\Delta$$

Reimann sum

Area of n rectangles

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} f(\zeta_k)(x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} f(a + k\Delta)\Delta \\
&= \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}
\end{aligned}
$$

Reimann integral

Taking limits

$$\int_a^b f(x)\,dx \;=$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right)\frac{b-a}{n}$$

Using n rectangles of equal width and evaluating at left endpoints

We get

$$\int_a^b f(x)\,dx \;=\; \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + (k-1)\frac{b-a}{n}\right)\frac{b-a}{n}$$

Using n rectangles of equal width and evaluating at left endpoints

Exercise

Properties of definite integrals

Properties of definite integrals

Linearity

For f , g integrable on $[a, b]$

$$\int_{x=a}^{x=b} [\alpha f(x) + \beta g(x)]\, dx \; = \; \alpha \int_{x=a}^{x=b} f(x)\, dx + \beta \int_{x=a}^{x=b} g(x)\, dx$$

What does this mean in terms of areas?

Example

Recall $\int_0^1 x dx = 1/2$

Evaluate

$$\int_0^1 3x \, dx$$

Example

Example

Integral of zero

$$\int_a^b 0\,dx = \int_a^b 0 f(x)\,dx = 0 \int_a^b f(x)\,dx = 0$$

Odd functions over symmetric intervals

If f is odd over $[-a, a]$ then

$$\int_{x=-a}^{x=+a} f(x)\,dx \;=\; 0$$

What does this mean in terms of areas?

Example

Evaluate

$$\int_{x=-123.4}^{x=+123.4} \sin^{135}(x)\,dx \;=\; 0$$

For $f(x) \geq 0$ on $[a, b]$

We have

$$\int_{x=a}^{x=b} f(x)\,dx \;\geq\; 0$$

What does this mean in terms of areas?

Example

What can you say about the integral?

$$\int_{x=1}^{x=136} \frac{x}{x^3+1}\,dx$$

Comparison

For $f(x) \leq g(x)$ on $[a, b]$

$$\int_{x=a}^{x=b} f(x)\,dx \;\leq\; \int_{x=a}^{x=b} g(x)\,dx$$

Example

Establish that

$$\int_{x=1}^{x=2} \frac{x}{1+x^2}\,dx \;\leq\; \int_{x=1}^{x=2} \frac{x+|\sin(x)|}{1+x^2}\,dx$$

Zero interval

For f integrable on $[a, b]$

$$\int_{x=a}^{x=a} f(x)\,dx \;=\; 0$$

Example

Example

$$\int_{x=2}^{x=2} |\sin^{136}(x) + \cos(x)|\, dx \;=\; 0$$

For f integrable on $[a, b]$ and $c \in [a, b]$

We have

$$\int_{x=a}^{x=b} f(x)\,dx = \int_{x=a}^{x=c} f(x)\,dx + \int_{x=c}^{x=b} f(x)\,dx$$

What does this mean in terms of areas?

Example

Define f by

$$f(x) = \begin{cases} -1 & x \geq 0 \\ +1 & x < 0 \end{cases}$$

Example contd

Evaluate

$$\int_{x=-1}^{x=+3} f(x)\, dx$$

Order of integration

For f integrable on $[a, b]$

$$\int_{x=b}^{x=a} f(x)\, dx \;=\; -\int_{x=a}^{x=b} f(x)\, dx$$

Example

We saw

$$\int_{x=0}^{x=1} x\,dx = \frac{1}{2}$$

so

$$\int_{x=1}^{x=0} x\,dx = ?$$

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