Inverse trig functions and their derivatives

S Sial

Dept of Mathematics
LUMS

Fall 2020-2021

Outline

Inverse sin function

Inverse sin function

Recall formula

Formula

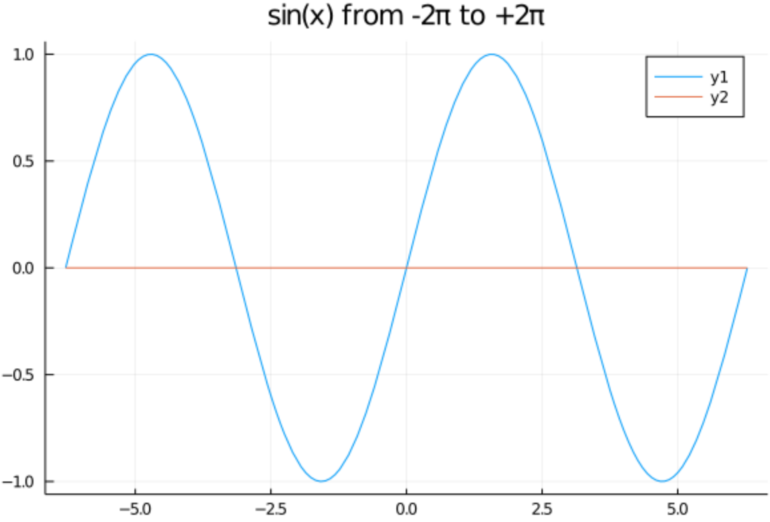
If $f : U \to V$ has derivative $f'(x) \neq 0$ and inverse $f^{-1} : V \to U$ then

$$\frac{d}{dy} f^{-1}(y) = \frac{1}{f'(x)}$$

where

$$y = f(x)$$

Consider plots of sin



Is sin : $\mathbb{R} \to \mathbb{R}$ 1-1 onto?

NOT 1-1, NOT onto, no inverse fn

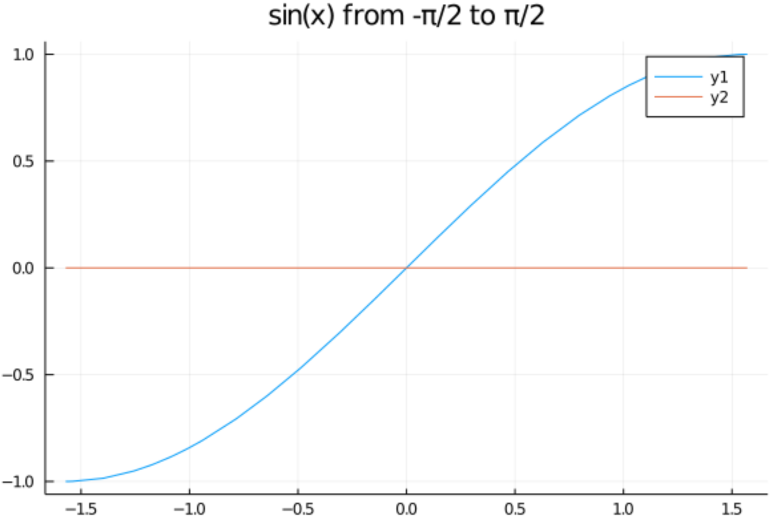
Redefine sin function

Defn

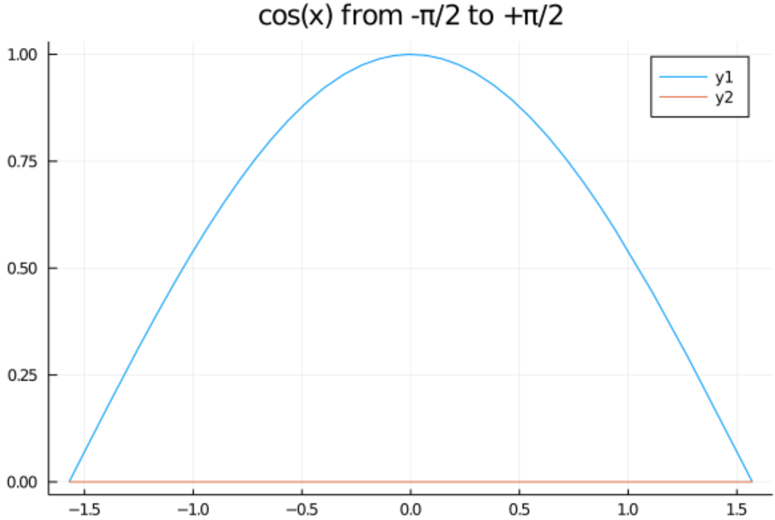
$$\sin : \left[-\frac{\pi}{2}\ ,\ \frac{\pi}{2}\right] \ \rightarrow\ [-1\ ,\ 1]$$

is 1-1 and onto so has an inverse

Consider plot of redefined sin



Consider plot of cos on the same interval



Consider plot of cos on the same interval

Notice

cos does not take negative values on the interval.

Inverse sin function

Can now talk about an inverse of the redefined sin

If

$$y = \sin(x)$$

Then

$$\sin^{-1}(y) = x$$

Inverse sin function

asin or arcsin or $\sin^{-1}$

$$\sin^{-1} : [-1,1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\sin^{-1}(\sin(x)) = x \quad \text{for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\sin(\sin^{-1}(y)) = y \quad \text{for } y \in [-1,1]$$

Derivative of asin

Apply rule for inverse fns

Let $y = \sin(x)$

$$\frac{d}{dy}\sin^{-1} y = \frac{1}{\frac{d}{dx}\sin(x)} = \frac{1}{\cos(x)}$$

Get rid of x

Apply trig identity

Had $y = \sin(x)$

And $\cos^2(x) + y^2 = \cos^2(x) + \sin^2(x) = 1$

So $\cos(x) = \pm\sqrt{1 - y^2}$

Is $\cos(x)$ negative or positive?

Consider values of x

$$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow \cos(x) \geq 0$$

$$\rightarrow \cos(x) = +\sqrt{1 - y^2}$$

Derivative of asin

Then

$$\frac{d}{dy}\sin^{-1} y = \frac{1}{\cos(x)} = \frac{1}{\sqrt{1-y^2}}$$

Finally

We get

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

Problem

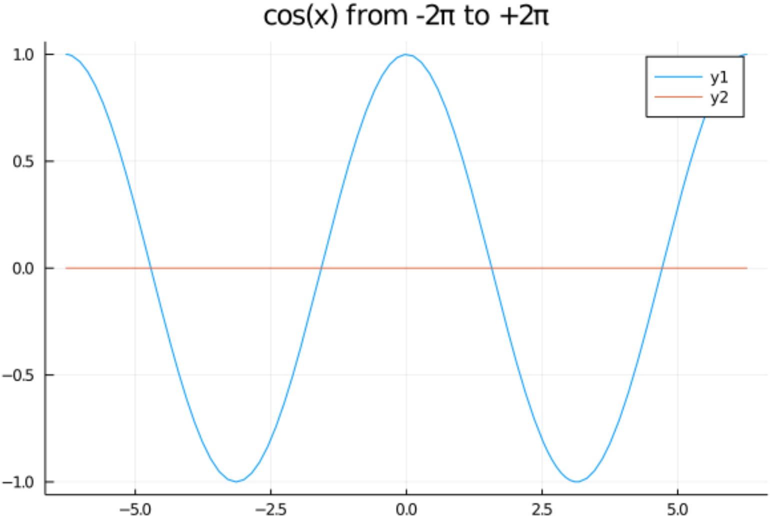
Differentiate

$$f(x) = 3\sin^{-1}\left(x^2 + x + 1\right)$$

Other inverse trig functions

Other inverse trig functions

Consider plot of cos



Is $\cos : \mathbb{R} \to \mathbb{R}$ 1-1 onto?

NOT 1-1, NOT onto, no inverse fn

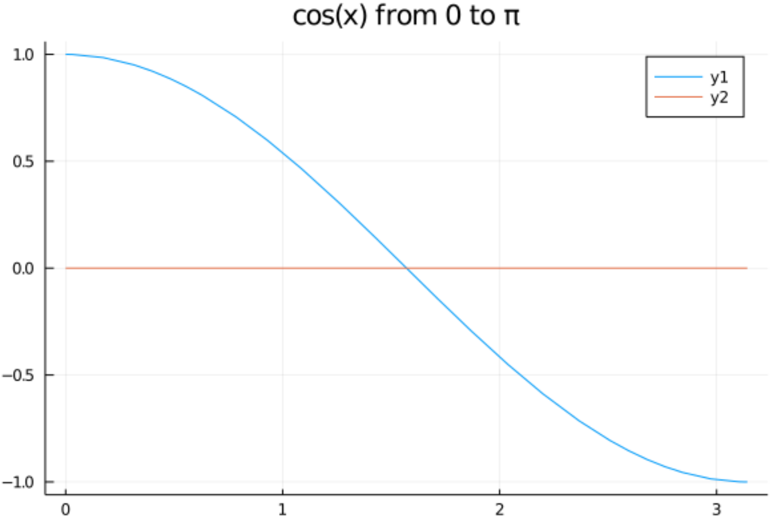
Redefine cos function

Defn

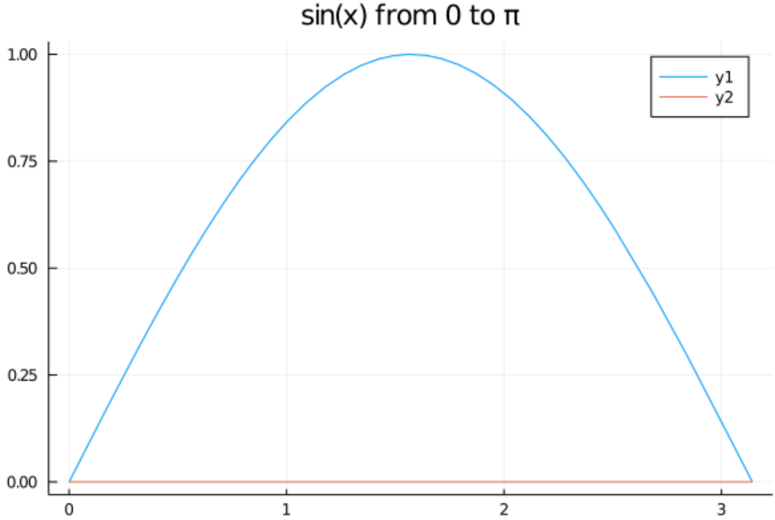
$$\cos : [0\ ,\ \pi] \ \rightarrow\ [-1\ ,\ 1]$$

is 1-1 and onto so has an inverse

Consider plot of redfined cos



Consider plot of sin on the same interval



Inverse cos function

Can now talk about an inverse fn

$$y = \cos(x)$$

$$x = \cos^{-1}(y)$$

Inverse cos function

acos or arccos or $\cos^{-1}$

$$\cos^{-1} : [-1, 1] \rightarrow [0, \; , \; \pi]$$

$$\cos^{-1}(\cos(x)) = x \qquad \text{for } x \in [0, \pi]$$

$$\cos(\cos^{-1}(y)) = y \qquad \text{for } y \in [-1, 1]$$

Derivative of acos

Apply rule for inverse fns

$$y = \cos(x)$$

$$\frac{d}{dy}\cos^{-1}(y) = \frac{1}{\frac{d}{dx}\cos(x)} = -\frac{1}{\sin(x)}$$

Get rid of x

Apply trig identity

Given $y = \cos(x)$

$$\sin^2(x) + y^2 = \sin^2(x) + \cos^2(x) = 1$$

Then

$$\sin(x) = \pm\sqrt{1 - y^2}$$

Is $\sin(x)$ negative or positive?

Consider values of x

On the interval, sin is never negative.

So

$$\sin(x) = \sqrt{1 - y^2}$$

Then

With this result for sin

$$\frac{d}{dy}\cos^{-1}(y) = -\frac{1}{\sin(x)} = -\frac{1}{\sqrt{1-y^2}}$$

Finally

We get

$$\frac{d}{dy}\cos^{-1}(y) = \frac{-1}{\sqrt{1-y^2}}$$

$$\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

Derivative of atan

Restrict domain of tan

$$\tan : (-\pi/2, \pi/2) \to \mathbb{R}$$

$$\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$$

Derivative of atan

Let $y(x) = \tan(x)$

$$
\begin{aligned}
\frac{d}{dy}\tan^{-1}(y) &= \frac{1}{\sec^2(x)} \\
&= \frac{1}{1+\tan^2(x)} \\
&= \frac{1}{1+y^2}
\end{aligned}
$$

Derivative of atan

Derivative of atan

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

Problem

Name a function f such that

$$\frac{d}{dx}f(x) = \frac{1}{1+4x^2}$$

Inverse cot

Restrict domain

$$\cot : (0, \pi) \rightarrow \mathbb{R}$$

$$\cot^{-1} : \mathbb{R} \rightarrow (0, \pi)$$

Derivative of acot

Derive this

$$\frac{d}{dx}\cot^{-1}(x) = \frac{-1}{1+x^2}$$

Inverse sec

Restrict domain

$$\sec : [0, \pi/2) \cup (\pi/2, \pi] \to (-\infty, -1] \cup [1, \infty)$$

$$\sec^{-1} : (-\infty, -1] \cup [1, \infty) \to [0, \pi/2) \cup (\pi/2, \pi]$$

Derivative of asec

Derive this

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$$

Inverse csc

Restrict domain

$$\csc : [-\pi/2, 0) \cup (0, \pi/2] \rightarrow (-\infty, -1] \cup [1, \infty)$$

$$\csc^{-1} : (-\infty, -1] \cup [1, \infty) \rightarrow [-\pi/2, 0) \cup (0, \pi/2]$$

Derivative of acsc

Derive this

$$\frac{d}{dx}\csc^{-1}(x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

Problems

Problems

Problem

Evaluate

$$\frac{d}{dx} x^3 \operatorname{asin}(x)$$

Problem

Differentiate

$$y = \frac{1 + \operatorname{atan} x}{1 - \operatorname{atan} x}$$

Problem

Problem

Let f be defined

$$f(x) = 2x + 10\cot^{-1}(x)$$

When is the tangent line to the graph of f horizontal?

Problem

Differentiate

$$\text{acot}(1/x) \; - \; \text{atan}(x)$$

What does the result imply?

Problem

Linearize atan near $x = 0$

$$\text{atan}(x) \approx x$$

for x small.