# Implicit differentiation, related rates, inverse functions

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# Outline

# Chain rule review

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# Recall chain rule

$$z = g(y(x))$$

$$\frac{dz}{dx} = \frac{dg}{dy} \cdot \frac{dy}{dx}$$

# Example

$$
\begin{aligned}
y(x) &= (ax^2 + bx + c)^{35} \\
\frac{dy}{dx} &= 35(ax^2 + bx + c)^{34} \cdot (2ax + b)
\end{aligned}
$$

# Implicit differentiation

Implicit differentiation

# Explicit functions

Have only dealt with explicit functions so far which define $y$ explicitly as a function of $x$

Examples

$$y(x) = x^2 + 7$$

$$y(x) = 3\sin(10x) + 9\cos(144x) + x^2 + x + 1$$

# Implicit functions

Dependent variable not defined explicitly in terms of independent variable

Examples of $y$ defined implicitly as fn of $x$

$$x^2 + y^2 = 1$$

$$\tan(xy) = 2$$

# Idea

Differentiate using the chain rule

$$\frac{d}{dx}f(y(x)) = f'(y)\cdot\frac{dy}{dx}$$

Then solve for the derivative

# Example

Find the eqn of the tangent line of the curve defined by

$$x^2 + y^2 = 1$$

At the point $(1/\sqrt{2}, 1/\sqrt{2})$

# Example contd

Think of $y$ as a fn of $x$

$$x^2 + y(x)^2 = 1$$

$$x^2 + [y(x)]^2 = 1$$

Differentiate using the chain rule

$$2x + 2y(x) \cdot y'(x) = 0$$

Solve for the derivative

$$y'(x) = -\frac{x}{y}$$

# Example contd

At the point of interest

$$y'(x = 1/\sqrt{2}) \;=\; -\left.\frac{x}{y}\right|_{(1/\sqrt{2},1/\sqrt{2})} \;=\; -\frac{1/\sqrt{2}}{1/\sqrt{2}} \;=\; -1$$

Recall eqn of tangent line at $(x_0, y_0)$

$$L(x) \;=\; y(x_0) + y'(x_0)(x - x_0)$$

# Example contd

## Eqn of tangent line

In our case, $x_0 = 1/\sqrt{2}$ and $y_0 = 1/\sqrt{2}$ and the derivative was $y'(x_0 = 1/\sqrt{2}) = -1$.

$$L(x) \;=\; y(x_0 = 1/\sqrt{2}) + y'(x_0 = 1/\sqrt{2})(x - x_0 = 1/\sqrt{2})$$

$$L(x) \;=\; \frac{1}{\sqrt{2}} - \left(x - \frac{1}{\sqrt{2}}\right)$$

# Example

Linearize $y$ at (1,1) given

$$xy^2 = 1$$

$y$ implicitly defined fn of $x$

$$x[y(x)]^2 = 1$$

Apply product rule

$$x \cdot \frac{d}{dx}[y(x)]^2 + \frac{d}{dx}x \cdot y^2 = 0$$

Apply chain rule

$$x \cdot 2y \cdot y' + 1 \cdot y^2 = 0$$

## Eqn of tangent line

$$L(x) = y(x_0) + y'(x_0)(x - x_0)$$

$$L(x) = 1 - \frac{1}{2}(x - 1)$$

# Example

Find $dy/dx$ given

$$y^2 + y + 1 = x^2 + x + 1$$

# Example contd

Take the deriv of both sides

$$\frac{d}{dx}\left(y^2 + y + 1\right) \;=\; \frac{d}{dx}(x^2 + x + 1)$$

$$\frac{d}{dy}\left(y^2 + y + 1\right) \cdot \frac{dy}{dx} \;=\; \frac{d}{dx}(x^2 + x + 1)$$

$$(2y + 1 + 0) \cdot \frac{dy}{dx} \;=\; 2x + 1 + 0$$

# Example contd

Simplify

$$(2y+1)\cdot\frac{dy}{dx} \;=\; 2x+1$$

Solve for the derivative

$$\frac{dy}{dx} \;=\; \frac{2x+1}{(2y+1)}$$