

Stochastic Portfolio Theory

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Main References:

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The Concept of "Returns"

Investors use the concept of returns to "quantify" or measure investment performance.

"Returns" refers to:

Profits, measured as a percentage of the investment amount:

What you've got minus what you had to begin with as a proportion of what you had to begin with.

Quantitative return measures are necessary to:

- Measure past performance.
- Express expectations about future performance.

Measuring Risk in Returns

Intuitively, risk in investment analysis refers to the possibility of not making the expected return.

We can measure risk by the standard deviation of the future return possibilities. We call this measure "volatility".

The larger the standard deviation, the greater the volatility or risk.

Arithmetic Return vs. Geometric Return

There are two mathematical indicators of returns on an investment: arithmetic means and geometric means.

Daily arithmetic returns are defined by

$$r_t = \frac{X_t - X_{t-1}}{X_{t-1}},$$

where X_t is the price of the asset at day t .

Yearly arithmetic returns are defined by

$$R = \frac{X_T - X_0}{X_0},$$

where X_0 and X_T are the prices of the asset at the first and last trading day of the year, respectively. We have that R may be written as

$$R = \frac{X_T}{X_0} - 1 = \frac{X_T}{X_{T-1}} \frac{X_{T-1}}{X_{T-2}} \dots \frac{X_1}{X_0} - 1 = \prod_{t=1}^T \frac{X_t}{X_{t-1}} - 1,$$

that is, it is not possible to describe the yearly arithmetic return as a function or sum of daily arithmetic returns.

The rate of return (expressed as a percent) is defined as:

$$100 \times \frac{X(t_1) - X(t_0)}{X(t_0)}$$

where X_t is the stock or bond price at time t .

Daily geometric returns are defined by

$$d_t = \log(X_t) - \log(X_{t-1}),$$

while *yearly geometric returns* are given by

$$D = \log(X_T) - \log(X_0),$$

We have that D may be written as

$$D = \log\left(\prod_{t=1}^T \frac{X_t}{X_{t-1}}\right) = \sum_{t=1}^T \log\left(\frac{X_t}{X_{t-1}}\right) = \sum_{t=1}^T d_t,$$

which means that yearly geometric returns are equal to the sum of daily geometric returns.

The relationship between yearly geometric and arithmetic returns is given by

$$D = \log(1 + R)$$

Arithmetic Return vs. Geometric Return continued

$$\text{arithmetic return (AR)} = \frac{X(t_1) - X(t_0)}{X(t_0)}$$

$$\text{geometric return (GR)} = \log \left(\frac{X(t_1)}{X(t_0)} \right)$$

By the rules for logarithms, the geometric return is also equal to $\log(X(t_1)) - \log(X(t_0))$. Now it is easy to see that geometric returns compound nicely:

$$\text{GR}_{t_0 \rightarrow t_1} + \text{GR}_{t_1 \rightarrow t_2} = \text{GR}_{t_0 \rightarrow t_2}$$

The geometric return gives a more accurate picture of the long term stock behaviour.

$$\text{instantaneous arithmetic return at time } t = \frac{dX(t)}{X(t)}$$

$$\text{instantaneous geometric return at time } t = d \log(X(t))$$

Stock price process

A stock price process X (a stock, for short) is a process that satisfies the following S.D.E:

$$dX(t) = X(t) \left[b(t)dt + \sum_{j=1}^m \sigma_j(t) dW_j(t) \right]$$

where

(W_1, \dots, W_m) is a m -dimensional Brownian motion

b rate of return of X

$\sigma_j(t)$ volatility of X : "sensitivity of the stock to the W_j "

$$X(t) = X_0 \exp \left[\int_0^t \gamma(s)ds + \int_0^t \sum_{j=1}^m \sigma_j(s) dW_j(s) \right]$$

where $X_0 > 0$ constant represents the initial value of the stock

and $\gamma(t) = b(t) - \frac{1}{2} \sum_{j=1}^m \sigma_j^2(t)$ is known as *growth rate* of X .

The logarithmic representation

$d \log X(t)$ represents the log-return (or continuous return) of X over the (short) time period dt .

$$d \log X(t) = \gamma(t) dt + \sum_{j=1}^m \sigma_j(t) dW_j(t),$$

where $\gamma(t) = b(t) - \frac{1}{2} \sum_{j=1}^m \sigma_j^2(t)$.

The higher is the variance rate, the larger will be the difference between the arithmetic and geometric returns.

Bachelier model

The first continuous-time stochastic model for a financial asset appeared in the thesis of Bachelier (1900). He proposed modelling the price of a stock as follows:

$$X_t = X_0 + \mu t + \sigma B_t$$

where X_0 , μ and σ are constants and B is a Brownian motion. The drawbacks of this model are that X_t can become negative and the relative returns are lower for higher stock prices.

Samuelson model

Samuelson (1965) introduced the more realistic model

$$X_t = X_0 \exp \left(\left\{ \mu - \frac{\sigma^2}{2} \right\} t + \sigma B_t \right)$$

where X_0 , μ and σ are constants and B is a Brownian motion. However, the Samuelson model also has deficiencies and up to now there have been many efforts to build better models.

Market

A *market* is a family $\mathcal{M} = \{X_1, \dots, X_n\}$ of stocks:

$$dX_i(t) = X_i(t) \left[b_i(t) dt + \sum_{j=1}^m \sigma_{ij}(t) dW_j(t) \right]$$

Standard assumptions:

- $m \geq n$ namely, that there exist at least as many factors of uncertainty in the market as there are stocks to hedge against them.

- $\int_0^t \|b(s)\| ds < \infty$ for all t , a.s.

- there exist $M, \epsilon > 0$, $M > \epsilon$ with

$$\epsilon \|v\|^2 \leq v^T \sigma(t) (\sigma(t))^T v \leq M \|v\|^2$$

for all $v \in \mathbb{R}^n$ and for all t , a.s. where matrix-valued process $\sigma(t) = (\sigma_{ij}(t))_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$

simplifying assumptions

We shall make certain simplifying assumptions, among them:

1. Companies do not merge or break up, and the total number of shares of a company remains constant. The list of companies in the market is fixed.
2. There are no dividend payments.
3. There are no transaction costs, taxes, or problems with the indivisibility of shares.

Portfolio

Portfolio in \mathcal{M} : measurable vector-valued process

$$\pi(t) = (\pi_1(t), \dots, \pi_n(t))$$

with

- π bounded a.s.
- $\pi_1(t) + \dots + \pi_n(t) = 1$ a.s.

The component π_i : the *proportions* or *weights*

A negative value for $\pi_i(t) \rightarrow$ short sale in the i th stocks.

Instantaneous return of the portfolio

Suppose we have a portfolio π and that $Z_\pi(t) > 0$ represents the value of an investment in π at time t . Then the amount invested in the i th stock X_i is

$$\pi_i(t) Z_\pi(t),$$

so, heuristically speaking, if the price of X_i changes by $dX_i(t)$, the induced change in the portfolio value is

$$\pi_i(t) Z_\pi(t) \frac{dX_i(t)}{X_i(t)}.$$

Hence the total change in the portfolio value at time t is

$$dZ_\pi(t) = \sum_{i=1}^n \pi_i(t) Z_\pi(t) \frac{dX_i(t)}{X_i(t)},$$

or, equivalently,

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^n \pi_i(t) \frac{dX_i(t)}{X_i(t)}. \quad (1)$$

Instantaneous return of the portfolio (2)

$$\begin{aligned} \frac{dZ_\pi(t)}{Z_\pi(t)} &= \sum_{i=1}^n \pi_i(t) b_i(t) dt \\ &\quad + \sum_{i=1}^n \sum_{j=1}^m \pi_i(t) \sigma_{ij}(t) dW_j(t). \end{aligned}$$

From Itô formula

$$d \log Z_\pi(t) = \gamma_\pi(t) dt + \sum_{i=1}^n \sum_{j=1}^m \pi_i(t) \sigma_{ij}(t) dW_j(t)$$

where $\gamma_\pi(t) = \sum_{i=1}^n \pi_i(t) \gamma_i(t) + \gamma_\pi^*(t)$ and

$$\gamma_\pi^*(t) = \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) a_{ii}(t) - \sum_{i=1}^n \sum_{j=1}^n \pi_i(t) \pi_j(t) a_{ij}(t) \right)$$

with $a_{ij}(t) = \sum_{k=1}^m \sigma_{ik}(t) \sigma_{jk}(t) = (\sigma \sigma^T)_{ij}$

γ_π^* is called the *excess growth rate* process of the portfolio.

$$\sigma_{\pi}^2(t) = \sum_{i=1}^n \sum_{j=1}^n \pi_i(t)\pi_j(t)a_{ij}(t)$$

is called the *portfolio variance* process

$$\gamma_{\pi}^*(t) = \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t)a_{ii}(t) - \sigma_{\pi}^2(t) \right)$$

the excess growth rate is one half the difference of the weighted average of the stock variances minus the portfolio variance.

Instantaneous logarithmic return of the portfolio

Let π be a portfolio and let Z_π be its value process. Then

$$d \log Z_\pi(t) = \sum_{i=1}^n \pi_i(t) d \log X_i(t) + \gamma_\pi^*(t) dt$$

so the instantaneous logarithmic return of the portfolio $d \log Z_\pi(t)$ is the weighted average of the instantaneous return of the component stocks, $d \log X_i(t)$, plus the excess growth rate.

Properties of the excess growth rate

Let π be a portfolio with nonnegative weights: $(\pi_i(t) \geq 0; 1 \leq i \leq n)$. Then

$$\gamma_{\pi}^*(t) = \gamma_{\pi}(t) - \sum_{i=1}^n \pi_i(t) \gamma_i(t) \geq 0$$

i.e. the portfolio growth rate always exceeds the weighted average of the growth rates of the component stocks.

In particular, when $\gamma_i = \gamma_j = \gamma$ for all i, j Then

$$\gamma_{\pi}(t) \geq \gamma(t)$$

Moreover, if for all $i, 0 \leq \pi_i(t) < 1$, for all $t \in [0, \infty)$, then

$$\gamma_{\pi}^*(t) > 0, \quad t \in [0, \infty)$$

That is for portfolios with non-negative weights, the excess growth rate is non-negative, and is positive unless the portfolio consists of a single stock.

The Markowitz Problem:

An investor might wish to find the highest rate of return possible for a given acceptable level of risk.

$$\begin{aligned} \text{minimize} \quad & \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t) \\ \text{subject to} \quad & \sum_{i=1}^n \pi_i(t)\alpha_i(t) \geq \alpha_0 \\ & \sum_{i=1}^n \pi_i(t) = 1 \text{ with } \pi_1(t), \dots, \pi_n(t) \geq 0 \end{aligned}$$

Markowitz showed how to formulate the problem of minimizing a portfolios variance subject to the constraint that its expected return equals a prescribed level as a quadratic program. Such an optimal portfolio is said to be **variance minimizing**.

The Market Portfolio

The portfolio μ with weights $\mu_1(t), \dots, \mu_n(t)$ defined by

$$\mu_i(t) = \frac{X_i(t)}{X_1(t) + \dots + X_n(t)}$$

is called the *market portfolio*: it invests in each company in proportion to its relative capitalization.

Let

$$Z(t) = X_1(t) + \dots + X_n(t) \quad \text{with} \quad x = Z(0)$$

Then we have:

$$\frac{dZ_\mu}{Z_\mu} = \sum_{i=1}^n \mu_i \frac{dX_i}{X_i} = \sum_{i=1}^n \frac{dX_i}{Z} = \frac{\sum_{i=1}^n dX_i}{Z} = \frac{dZ}{Z}$$

The value Z_μ of the market portfolio satisfies:

$$Z_\mu(t) = \frac{z}{x} \times (X_1(t) + \dots + X_n(t))$$

i.e reflects the total capitalization of the market in proportion to its initial investment $z > 0$. **We shall let μ exclusively represent the market portfolio and Z_μ its value process**

Relative Return

For portfolios π and η , the *relative return* process of π versus η is defined by

$$\log\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right)$$

and we have:

$$d \log\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) = \sum_{i=1}^n \pi_i(t) d \log\left(\frac{X_i(t)}{Z_\eta(t)}\right) + \gamma_\pi^*(t) dt.$$

In particular, when $\eta = \mu$, the market portfolio:

$$d \log\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) = \sum_{i=1}^n \pi_i(t) d \log(\mu_i(t)) + \gamma_\pi^*(t) dt.$$

Hence, we can analyze the relative portfolio performance in terms of the changing market weights and the excess growth rate.

Long-term behavior of a portfolio of stocks

Portfolio growth rate determines long-term portfolio behavior:

For any portfolio π in \mathcal{M}

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left(\log Z_{\pi}(T) - \int_0^T \gamma_{\pi}(t) dt \right) = 0 \quad a.s.$$

Since *single stocks* can be considered portfolios, it follows:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left(\log X(T) - \int_0^T \gamma_{\pi}(t) dt \right) = 0 \quad a.s.$$

Coherence: a structural condition on the market

The market \mathcal{M} is *coherent* if for $i = 1, \dots, n$

$$\lim_{t \rightarrow \infty} t^{-1} \log \mu_i(t) = 0, \quad a.s.$$

Interpretation: none of the stocks declines too rapidly.

equivalent definition

(i) \mathcal{M} is coherent

$$(ii) \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\gamma_i(t) - \gamma_\mu(t)) dt = 0 \text{ for } i = 1, \dots, n$$

$$(iii) \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\gamma_i(t) - \gamma_j(t)) dt = 0 \text{ for } i, j = 1, \dots, n$$

in a coherent market, the time-average difference between the growth rates of any two stocks will be zero.

example of coherent market

one in which all the stocks have the **same growth rate process**

Stock market diversity

Heuristically speaking one could say that a market is diverse if the capital is not concentrated on a few big companies.

The market \mathcal{M} is *diverse* on $[0, T]$ if there exists a number $\delta \in (0, 1)$ such that

$$\mu_{max}(t) := \max_{1 \leq i \leq n} \mu_i(t) \leq 1 - \delta,$$

and is *weakly divers* on $[0, T]$ if there exists a number $\delta \in (0, 1)$ such that

$$\frac{1}{T} \int_0^T \mu_{max}(t) dt \leq 1 - \delta,$$

Entropy as a measure of market diversity

The *entropy* function (Shannon 1948) S is defined by

$$S = - \sum_{i=1}^n x_i \log x_i$$

for all x in

$$\Delta^n = \{x \in \mathbf{R}^n : \sum_{i=1}^n x_i = 1; 0 < x_i < 1, i = 1, \dots, n\}$$

The market portfolio

Let μ be the market portfolio. The *market entropy* $S(\mu)$ process is defined by

$$S(\mu(t)) = - \sum_{i=1}^n \mu_i(t) \log \mu_i(t)$$

The portfolio π with weights defined by

$$\pi_i(t) = \frac{-\mu_i(t) \log \mu_i(t)}{S(\mu(t))}$$

is called the *entropy-weighted* portfolio.

The ratio

$$\frac{\pi_i(t)}{\mu_i(t)} = -\frac{\log \mu_i(t)}{\mathbf{S}(\mu(t))}$$

decreases with increasing $\mu_i(t)$: Hence π is less concentrated than μ in those stocks with the highest market weights.

The relative return of the entropy portfolio is related to the variation of the market entropy:

$$d \log \mathbf{S}(\mu(t)) = d \log \frac{Z_\pi(t)}{Z_\mu(t)} - \frac{\gamma_\mu^*(t)}{\mathbf{S}(\mu(t))} dt$$

From this equation, it can be shown that in a nondegenerate and diverse market, for a sufficiently large number T

$$\boxed{\frac{Z_\pi(T)}{Z_\pi(0)} > \frac{Z_\mu(T)}{Z_\mu(0)} \quad a.s.}$$

This means that, with probability one, the entropy-weighted portfolio has return above of the market portfolio

Portfolio Generating Functions

Let S be a positive continuous function defined on

$$\Delta^n = \{x \in \mathbf{R}^n : \sum_{i=1}^n x_i = 1; 0 < x_i < 1, i = 1, \dots, n\}$$

Let π be a portfolio.

S *generates* π if there exists a measurable process of bounded variation Θ such that

$$\log\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) = \log S(\mu(t)) + \Theta(t)$$

- S is called *generating function* of the portfolio π .
- $\Theta(t)$ is called *the drift process* corresponding to S .

Proposition *Let S generate the portfolio π with drift process Θ , and suppose that*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log S(\mu(t)) = 0, \quad a.s. \quad (\star)$$

Then,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \left(\int_0^T \gamma_\pi(t) dt - \int_0^T \gamma_\mu(t) dt - \Theta(T) \right) = 0$$

Condition (\star) will be satisfied, for example, if S is bounded on Δ^n , this is the condition that pertains in most applications.

Functionally Generated Portfolios: Main Theorem (Fernholz 1999)

Let \mathcal{M} be a market of stocks X_1, \dots, X_n . Let S be a function defined on a neighborhood U of Δ^n . Suppose that there exists a positive C^2 function S defined on U .

Then S generates the portfolio π such that for $k = 1, \dots, n$,

$$\pi_i(t) = \left(D_i \log S(\mu(t)) + 1 - \sum_{j=1}^n \mu_j(t) D_j \log S(\mu(t)) \right) \mu_i(t),$$

for all $t \in [0, T]$, a.s.,

with a drift process Θ that satisfies

$$d\Theta(t) = \frac{-1}{2S(\mu(t))} \sum_{i,j=1}^n D_{ij} S(\mu(t)) \mu_i(t) \mu_j(t) \tau_{ij}(t) dt$$

where

$$\tau_{ij}(t) dt = d \langle \log \mu_i, \log \mu_j \rangle_t$$

Remark: The weights π_1, \dots, π_n depend only on the market weights, not on the covariance structure of the market.

Examples of generating functions and the portfolios they generate:

1. $S(x) = 1$ generates the market portfolio μ with $\Theta(t) = 0$.
2. $S(x) = c_1x_1 + \cdots + c_nx_n$, where c_1, \dots, c_n are constants, generates the buy-and-hold portfolio which holds c_i of the capitalization of the the i -th stock. Here $\Theta(t) = 0$. This type of portfolio is commonly held by investors, at least in a piecewise manner.

Not all portfolios are functionally generated

The No-Arbitrage Hypothesis

Denote by V_t the portfolio value at time t . An arbitrage opportunity is the existence of a self-financing trading strategy and a time t such that $V_0 = 0$, $V_t \geq 0$ a.s. and $P[V_t > 0] > 0$ (or, equivalently, $EV_t > 0$.)

It is axiomatic that arbitrage cannot exist in the market, so no mathematical model should permit arbitrage opportunities.

Relative Arbitrage

Given any two portfolio rules π, ρ and a real constant $T > 0$, we shall say that $\pi(\cdot)$ is an arbitrage opportunity relative to $\rho(\cdot)$ over the time-horizon $[0; T]$ if there exists a real constant $q = q_{\pi, \rho, T} > 0$ such that

$$\begin{aligned} P\left(\frac{Z_{\pi}(t)}{Z_{\rho}(t)} \geq q, \forall 0 \leq t \leq T\right) &= 1, \\ P[Z_{\pi}(T) \geq Z_{\rho}(T)] &= 1, \\ P[Z_{\pi}(T) > Z_{\rho}(T)] &> 0 \end{aligned}$$

hold, whenever the two portfolio rules start with the same initial fortune $Z_{\pi}(0) = Z_{\rho}(0) = z \in (0, \infty)$.

Sufficient Conditions for Relative Arbitrage

We show that if the cumulative excess growth (i.e., the integral of the excess growth rate) of the market portfolio over a given period of time is sufficiently large, then relative arbitrage exists in the market over that period.

Measure of Diversity

Interests:

1. Market diversity is an **observable** characteristic of equity markets that is amenable to stochastic analysis.
2. Measures of diversity can be used to construct portfolios with desirable investment characteristics.

A positive C^2 function defined on an open neighborhood of Δ^n is a *measure of diversity* if it is symmetric and concave.

A portfolio generated by a measure of diversity is called a *diversity-weighted* portfolio, and its proportions are called *diversity weights*.

Proposition

Suppose that S is a measure of diversity that generates a portfolio π with drift Θ . Then

(i) Θ is a.s. nondecreasing

(ii) $\mu_i(t) \geq \mu_j(t)$ implies that $\pi_j(t)/\mu_j(t) \geq \pi_i(t)/\mu_i(t)$

(ii) shows that the weight ratios $\pi_i(t)/\mu_i(t)$ decrease with increasing market weight. Hence, if a stock's market weight increases, i.e., the stock goes up relative to the market, then the portfolio π will sell some (fractional) shares of that stock.

Examples of measure of diversity

1. Entropy function

$$S = - \sum_{i=1}^n x_i \log x_i$$

2. For $0 < p < 1$, let

$$\mathbf{D}_p(x) = \left(\sum_{i=1}^n x_i^p \right)^{1/p}$$

The portfolio generated by \mathbf{D}_p has weights

$$\pi_i(t) = \frac{\mu_i^p(t)}{(\mathbf{D}_p(\mu(t)))^p} = \frac{\mu_i^p(t)}{\sum_{j=1}^n \mu_j^p(t)}$$

The drift process:

$$d\Theta(t) = (1 - p)\gamma_{\pi}^*(t)$$

Note that as $p \rightarrow 1$, π approaches the market portfolio.

Local times of continuous semimartingale

If $X = (X_t, t \geq 0)$ is a continuous semimartingale, then by **Tanaka's formula**, for every $a \in \mathbf{R}$, we have

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a) dX_s + L_t^a(X)$$

where $\operatorname{sgn} = -1\{(-\infty, 0]\} + 1\{(0, \infty)\}$

and

$L_t^a(X)$ continuous increasing process called the local time of X at a .

Decomposition of Ranked Continuous Semimartingales

Let X_1, \dots, X_n be continuous semimartingales. For $1 \leq k \leq n$, the k -th rank process of X_1, \dots, X_n is defined by

$$X^{(k)} = \max_{i_1 < \dots < i_k} \min(X_{i_1}, \dots, X_{i_k})$$

where $1 \leq i_1$ and $i_k \leq n$.

Note that,

$$\max_{1 \leq i \leq n} X_i(t) = X^{(1)}(t) \geq X^{(2)}(t) \geq \dots \geq X^{(n)}(t) = \min_{1 \leq i \leq n} X_i(t)$$

so that at any given time, the values of the rank processes represent the values of the original processes arranged in descending order (i.e. the (reverse) order statistics).

Proposition

Let X_1, \dots, X_n be continuous semimartingales. For $k \in \{1, 2, \dots, n\}$, Let $u(k) = (u_t(k), t \geq 0) : \Omega \times [0, \infty[\rightarrow \{1, 2, \dots, n\}$ be any predictable process with the property:

$$X_t^{(k)} = X_{u_t(k)}(t)$$

Then the k -th rank processes $X^{(k)}, k = 1, \dots, n$, are continuous semimartingales and we have:

$$\begin{aligned} X^{(k)}(t) &= X^{(k)}(0) + \sum_{i=1}^n \int_0^t 1_{\{u_s(k)=i\}} dX_i(s) \\ &+ \frac{1}{2} \sum_{i=1}^n \int_0^t 1_{\{u_s(k)=i\}} d_s L_s^0((X^{(k)} - X_i)^+) \\ &- \frac{1}{2} \sum_{i=1}^n \int_0^t 1_{\{u_s(k)=i\}} d_s L_s^0((X^{(k)} - X_i)^-). \end{aligned}$$

where $L_t^0(X)$ is the local time of the continuous semimartingale X at 0.

Theorem:

Let X_1, \dots, X_n be continuous semimartingales. Then the k -th rank processes $X^{(k)}, k = 1, \dots, n$, are continuous semimartingales and we have:

$$\begin{aligned} X^{(k)}(t) = & X^{(k)}(0) + \sum_{i=1}^n \int_0^t \mathbf{1}_{\{u_s(k)=i\}} dX_i(s) \\ & + \frac{1}{2} \sum_{i=k+1}^n \int_0^t \frac{1}{N_s(k)} d_s L_s^0(X^{(k)} - X^{(i)}) \\ & - \frac{1}{2} \sum_{i=1}^{k-1} \int_0^t \frac{1}{N_s(k)} d_s L_s^0(X^{(i)} - X^{(k)}). \end{aligned}$$

where $L_t^0(X)$ is the local time of the continuous semimartingale X at 0 and

$$N_t(k) = \sum_{j=1}^n \mathbf{1}_{\{X_t^{(j)} = X_t^{(k)}\}}$$

Portfolio Generated by Functions of Ranked Market Weights:

The ranked family of market weights $\{\mu_{(1)}(t), \dots, \mu_{(n)}(t)\}$ will be called the *capital distribution* of the market at time t .

Theorem

Let \mathcal{M} be a market of stocks X_1, \dots, X_n . Let S be a function defined on a neighborhood U of Δ^n . Suppose that there exists a positive C^2 function S defined on U .

Then S generates the portfolio π such that for $k = 1, \dots, n$,

$$\pi_{u_t(k)}(t) = \left[D_k \log S(\mu_{(\cdot)}(t)) + 1 - \sum_{j=1}^n \mu_{(j)}(t) D_j \log S(\mu_{(\cdot)}(t)) \right] \mu_{(k)}(t),$$

with a drift process Θ that satisfies

$$\begin{aligned}
d\Theta(t) = & \\
& \frac{-1}{2\mathbf{S}(\mu(t))} \sum_{i,j=1}^n D_{ij} S(\mu_{(\cdot)}(t)) \mu_{(i)}(t) \mu_{(j)}(t) \tau_{(ij)}(t) dt \\
& - \frac{1}{2} \sum_{k=1}^n \frac{\pi_{u_t(k)}(t)}{N_t(k)} \sum_{i=k+1}^n dL_t^0(\log(\mu_{(k)}) - \log(\mu_{(i)})) \\
& + \frac{1}{2} \sum_{k=1}^n \frac{\pi_{u_t(k)}(t)}{N_t(k)} \sum_{i=1}^{k-1} dL_t^0(\log(\mu_{(i)}) - \log(\mu_{(k)}))
\end{aligned}$$

Where

$$\tau_{(ij)}(t) = \tau_{u_t(i)u_t(j)}(t)$$