One-factor Interest Rate Modeling

Lecture Notes to Accompany Wilmott

In this lecture. . .

- stochastic models for interest rates

- how to derive the bond pricing equation for many fixed-income products

- the structure of many popular interest rate models

Introduction

In this lecture we see the ideas behind modeling interest rates using a single source of randomness. This is **one-factor interest rate modeling**.

- The model will allow the short-term interest rate, the spot rate, to follow a random walk.

This model leads to a parabolic partial differential equation for the prices of bonds and other interest rate derivative products.

The ‘spot rate’ that we will be modeling is a very loosely-defined quantity, meant to represent the yield on a bond of infinitesimal maturity.

In practice one should take this rate to be the yield on a liquid finite-maturity bond, say one of one month.

Bonds with one *day* to expiry do exist but their price is not necessarily a guide to other short-term rates.

Stochastic interest rates

Since we cannot realistically forecast the future course of an interest rate, it is natural to model it as a random variable.

- We are going to model the behaviour of r, the interest rate received by the shortest possible deposit.

From this we will see the development of a model for all other rates. The interest rate for the shortest possible deposit is commonly called the **spot interest rate**.

Let us suppose that the interest rate r is governed by a stochastic differential equation of the form

- $$dr = u(r,t)\,dt + w(r,t)\,dX.$$

The functional forms of $u(r,t)$ and $w(r,t)$ determine the behaviour of the spot rate r. For the present we will not specify any particular choices for these functions.

The bond pricing equation for the general model

When interest rates are stochastic a bond has a price of the form $V(r,t;T)$.

- Pricing a bond presents new technical problems, and is in a sense harder than pricing an option since *there is no underlying asset with which to hedge*.

We are therefore not modeling a *traded* asset; the traded asset (the bond, say) is a derivative of our independent variable r.

- The only way to construct a hedged portfolio is by hedging one bond with a bond of a different maturity.

We set up a portfolio containing two bonds with different maturities T_1 and T_2.

The bond with maturity T_1 has price $V_1(r, t; T_1)$ and the bond with maturity T_2 has price $V_2(r, t; T_2)$.

We hold one of the former and a number $-\Delta$ of the latter.

We have

$$\Pi = V_1 - \Delta V_2.$$

The change in this portfolio in a time dt is given by

$$d\Pi = \frac{\partial V_1}{\partial t}dt + \frac{\partial V_1}{\partial r}dr + \tfrac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2}dt - \Delta\left(\frac{\partial V_2}{\partial t}dt + \frac{\partial V_2}{\partial r}dr + \tfrac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}dt\right),$$

where we have applied Itô's lemma to functions of r and t.

Which of these terms are random?

Once you've identified them you'll see that the choice

$$\Delta = \frac{\partial V_1}{\partial r}\bigg/\frac{\partial V_2}{\partial r}$$

eliminates all randomness in $d\Pi$. This is because it makes the coefficient of dr zero.

We then have

$$d\Pi = \left(\frac{\partial V_1}{\partial t} + \tfrac{1}{2} w^2 \frac{\partial^2 V_1}{\partial r^2} - \left(\frac{\partial V_1}{\partial r} \Big/ \frac{\partial V_2}{\partial r} \right) \left(\frac{\partial V_2}{\partial t} + \tfrac{1}{2} w^2 \frac{\partial^2 V_2}{\partial r^2} \right) \right) dt$$

$$= r\Pi \, dt = r \left(V_1 - \left(\frac{\partial V_1}{\partial r} \Big/ \frac{\partial V_2}{\partial r} \right) V_2 \right) dt,$$

where we have used arbitrage arguments to set the return on the portfolio equal to the risk-free rate. This risk-free rate is just the spot rate.

Collecting all V_1 terms on the left-hand side and all V_2 terms on the right-hand side we find that

$$\frac{\frac{\partial V_1}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V_1}{\partial r^2} - r V_1}{\frac{\partial V_1}{\partial r}} = \frac{\frac{\partial V_2}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V_2}{\partial r^2} - r V_2}{\frac{\partial V_2}{\partial r}}.$$

At this point the distinction between the equity and interest-rate worlds starts to become apparent.

This is *one* equation in *two* unknowns.

Fortunately, the left-hand side is a function of T_1 but not T_2 and the right-hand side is a function of T_2 but not T_1. The only way for this to be possible is for both sides to be independent of the maturity date.

Dropping the subscript from V, we have

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} - rV}{\frac{\partial V}{\partial r}} = a(r,t)$$

for some function $a(r,t)$.

We shall find it convenient to write

$$a(r,t) = w(r,t)\lambda(r,t) - u(r,t);$$

for a given $u(r,t)$ and non-zero $w(r,t)$ this is always possible.

The bond pricing equation is therefore

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0. \qquad (1)$$

The final condition corresponds to the payoff on maturity and so for a zero-coupon bond

$$V(r, T; T) = 1.$$

It is easy to incorporate coupon payments into the model. If an amount $K(r,t)dt$ is received in a period dt then

$$\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV + K(r,t) = 0.$$

When this coupon is paid discretely, arbitrage considerations lead to jump condition

$$V(r, t_c^-; T) = V(r, t_c^+; T) + K(r, t_c),$$

where a coupon of $K(r, t_c)$ is received at time t_c.

What is the market price of risk?

Imagine that you hold an unhedged position in one bond with maturity date T. In a time-step dt this bond changes in value by

$$dV = w\frac{\partial V}{\partial r}dX + \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r}\right)dt.$$

This may be written as

$$dV = w\frac{\partial V}{\partial r}dX + \left(w\lambda\frac{\partial V}{\partial r} + rV\right)dt,$$

or

$$dV - rV\,dt = w\frac{\partial V}{\partial r}(dX + \lambda\,dt).$$

This expression contains a deterministic term in dt and a random term in dX.

The deterministic term may be interpreted as the excess return above the risk-free rate for accepting a certain level of risk.

- In return for taking the risk the portfolio profits by $\lambda\,dt$ per unit of risk, dX. The function λ is called the **market price of risk**.

Interpreting the market price of risk, and risk neutrality

The bond pricing equation contains references to the functions $u - \lambda w$ and w. The former is the coefficient of the first-order derivative with respect to the spot rate, and the latter appears in the coefficient of the diffusive, second-order derivative.

The four terms in the equation represent, in order as written,

- time decay,
- diffusion,
- drift and
- discounting.

The equation is similar to the backward equation for a probability density function except for the final discounting term.

- As such we can interpret the solution of the bond pricing equation as the expected present value of all cashflows.

As with equity options, this expectation is not with respect to the *real* random variable, but instead with respect to the *risk-neutral* variable.

- There is this difference because the drift term in the equation is not the drift of the real spot rate u, but the drift of another rate, called the **risk-neutral spot rate**. This rate has a drift of $u - \lambda w$.

When pricing interest rate derivatives (including bonds) it is important to model, and price, using the risk-neutral rate. This rate satisfies

$$dr = (u - \lambda w)dt + w\, dX.$$

We need the new market-price-of-risk term because our modeled variable, r, is not traded.

- If we set λ to zero then any results we find are applicable to the *real* world. If, for example, we want to find the real distribution of the spot interest rate at some time in the future then we would solve a Fokker–Planck equation with the real, and not the risk-neutral, drift.

Tractable models and solutions of the bond pricing equation

We have built up the bond pricing equation for an arbitrary model. That is, we have not specified the risk-neutral drift, $u - \lambda w$, or the volatility, w. How can we choose these functions to give us a good model?

First of all, a simple lognormal random walk would *not* be suitable for r, since it would predict exponentially rising or falling rates. This rules out the equity price model as an interest rate model.

Let us examine some choices for the risk-neutral drift and volatility that lead to tractable models, that is, models for which the solution of the bond pricing equation for zero-coupon bonds can be found analytically.

We will discuss these models and see what properties we like or dislike.

For example, assume that $u - \lambda w$ and w take the form

$$u(r,t) - \lambda(r,t)w(r,t) = \eta(t) - \gamma(t)r,$$

$$w(r,t) = \sqrt{\alpha(t)r + \beta(t)}.$$

By suitably restricting these time-dependent functions, we can ensure that the random walk for r has the following nice properties:

- **Positive interest rates**: Except for a few pathological cases interest rates are positive. With the above model the spot rate can be bounded below by a positive number if $\alpha(t) > 0$ and $\beta \leq 0$. The lower bound is $-\beta/\alpha$. Note that r can still go to infinity, but with probability zero.

- **Mean reversion**: Examining the drift term, we see that for large r the (risk-neutral) interest rate will tend to decrease towards the mean, which may be a function of time. When the rate is small it will move up on average.

We also want the lower bound to be non-attainable, we don't want the spot interest rate to get forever stuck at the lower bound or have to impose further conditions to say how fast the spot rate moves away from this value.

This requirement means that

$$\eta(t) \geq -\beta(t)\gamma(t)/\alpha(t) + \alpha(t)/2.$$

We have chosen u and w in the stochastic differential equation for r to take special functional forms for a very special reason. With these choices the solution for the zero-coupon bond is of the simple form

- $$Z(r, t; T) = e^{A(t;T)-rB(t;T)}. \quad (2)$$

We are going to be looking at zero-coupon bonds specifically for a while, hence the change of our notation from V, meaning many interest rate products, to the very specific Z for zero-coupon bonds.

- The model with all of α, β, γ and η non-zero is the most general stochastic differential equation for r which leads to a solution of the form (2).

Let's see how this works.

Substitute (2) into the bond pricing equation (1). This gives

$$\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} + \tfrac{1}{2}w^2 B^2 - (u - \lambda w)B - r = 0. \qquad (3)$$

Some of these terms are functions of t and T (i.e. A and B) and others are functions of r and t (i.e. u and w).

Differentiating (3) with respect to r gives

$$-\frac{\partial B}{\partial t} + \tfrac{1}{2}B^2\frac{\partial}{\partial r}(w^2) - B\frac{\partial}{\partial r}(u - \lambda w) - 1 = 0.$$

Differentiate again with respect to r and divide through by B:

$$\tfrac{1}{2}B\frac{\partial^2}{\partial r^2}(w^2) - \frac{\partial^2}{\partial r^2}(u - \lambda w) = 0.$$

In this, only B is a function of T, therefore we must have

$$\frac{\partial^2}{\partial r^2}(w^2) = 0, \quad \frac{\partial^2}{\partial r^2}(u - \lambda w) = 0.$$

The equations for A and B are

$$\frac{\partial A}{\partial t} = \eta(t)B - \tfrac{1}{2}\beta(t)B^2 \tag{4}$$

and

$$\frac{\partial B}{\partial t} = \tfrac{1}{2}\alpha(t)B^2 + \gamma(t)B - 1. \tag{5}$$

In order to satisfy the final data that $Z(r, T; T) = 1$ we must have

$$A(T; T) = 0 \quad \text{and} \quad B(T; T) = 0.$$

Solution for constant parameters

The solution for arbitrary α, β, γ and η is found by integrating the two ordinary differential equations (4) and (5).

Generally speaking, though, when these parameters are time dependent this integration cannot be done explicitly.

But in some special cases this integration *can* be done explicitly.

The simplest case is when α, β, γ and η are all constant, in which case

$$\frac{\alpha}{2}A = a\psi_2 \log(a-B) + (\psi_2 + \tfrac{1}{2}\beta) b \log((B+b)/b) - \tfrac{1}{2}B\beta - a\psi_2 \log a,$$

and

$$B(t;T) = \frac{2(e^{\psi_1(T-t)}-1)}{(\gamma+\psi_1)(e^{\psi_1(T-t)}-1)+2\psi_1},$$

where

$$b, a = \frac{\pm\gamma + \sqrt{\gamma^2+2\alpha}}{\alpha},$$

and

$$\psi_1 = \sqrt{\gamma^2+2\alpha} \quad \text{and} \quad \psi_2 = \frac{\eta - a\beta/2}{a+b}.$$

When all four of the parameters are constant it is obvious that both A and B are functions of only the one variable $\tau = T - t$, and not t and T individually; this would not necessarily be the case if any of the parameters were time dependent.

A wide variety of yield curves can be predicted by the model. As $\tau \to \infty$,

$$B \to \frac{2}{\gamma + \psi_1}$$

and the yield curve Y has long term behaviour given by

$$Y \to \frac{2}{(\gamma + \psi_1)^2} \left(\eta(\gamma + \psi_1) - \beta \right).$$

Thus for constant and fixed parameters the model leads to a fixed long-term interest rate, independent of the spot rate.

The probability density function, $P(r,t)$, for the risk-neutral spot rate satisfies

$$\frac{\partial P}{\partial t} = \tfrac{1}{2}\frac{\partial^2}{\partial r^2}(w^2 P) - \frac{\partial}{\partial r}((u - \lambda w)P).$$

In the long term this settles down to a distribution, $P_\infty(r)$, that is independent of the *initial* value of the rate. This distribution satisfies the ordinary differential equation

- $$\tfrac{1}{2}\frac{d^2}{dr^2}(w^2 P_\infty) = \frac{d}{dr}((u - \lambda w)P_\infty).$$

The solution of this for the general affine model with constant parameters is

$$P_\infty(r) = \frac{\left(\frac{2\gamma}{\alpha}\right)^k}{\Gamma(k)} \left(r + \frac{\beta}{\alpha}\right)^{k-1} e^{-\frac{2\gamma}{\alpha}\left(r+\frac{\beta}{\alpha}\right)}$$

where

$$k = \frac{2\eta}{\alpha} + \frac{2\beta\gamma}{\alpha^2}$$

and $\Gamma(\cdot)$ is the gamma function. The boundary $r = -\beta/\alpha$ is non-attainable if $k > 1$. The mean of the steady-state distribution is

$$\frac{\alpha k}{2\gamma} - \frac{\beta}{\alpha}.$$

Named models

There are many interest rate models, associated with the names of their inventors.

- Vasicek
- Cox, Ingersoll & Ross
- Ho & Lee
- Hull & White

and many more.

Vasicek

The Vasicek model takes the form

- $$dr = (\eta - \gamma r)dt + \beta^{1/2} dX.$$

The value of a zero-coupon bond is given by

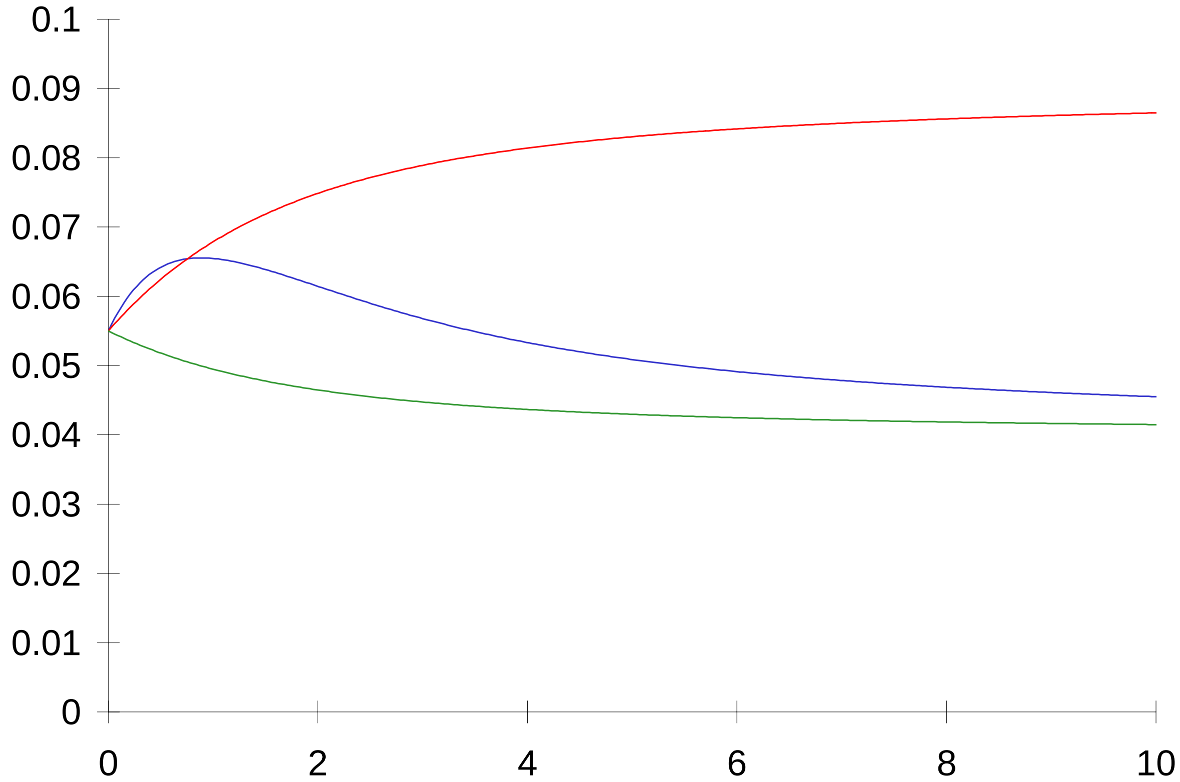
$$e^{A(t;T) - rB(t;T)}$$

where

$$B = \frac{1}{\gamma}(1 - e^{-\gamma(T-t)})$$

and

$$A = \frac{1}{\gamma^2}(B(t;T) - T + t)(\eta\gamma - \tfrac{1}{2}\beta) - \frac{\beta B(t;T)^2}{4\gamma}.$$

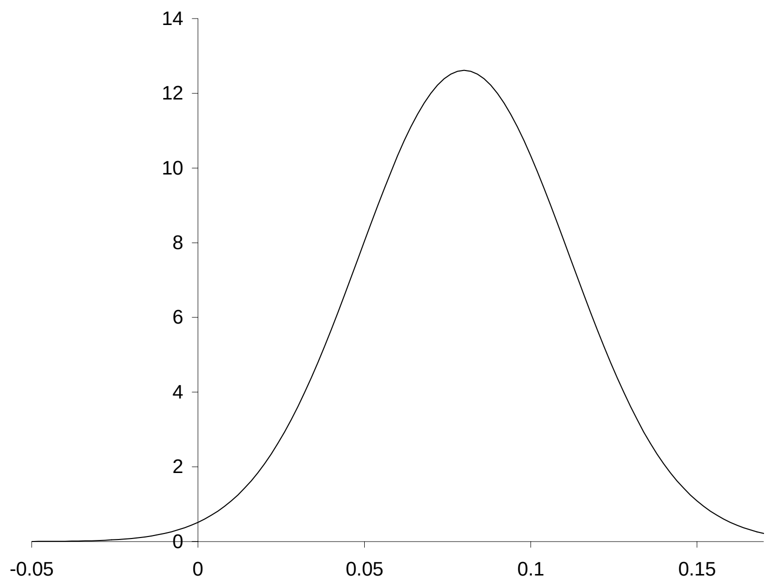


Three types of yield curve given by the Vasicek model.

The steady-state probability density function for the Vasicek model is

- $$P_\infty(r) = \sqrt{\frac{\gamma}{\beta\pi}} e^{-\frac{\gamma}{\beta}\left(r - \frac{\eta}{\gamma}\right)^2}.$$

Thus, in the long run, the spot rate is Normally distributed.



Cox, Ingersoll & Ross

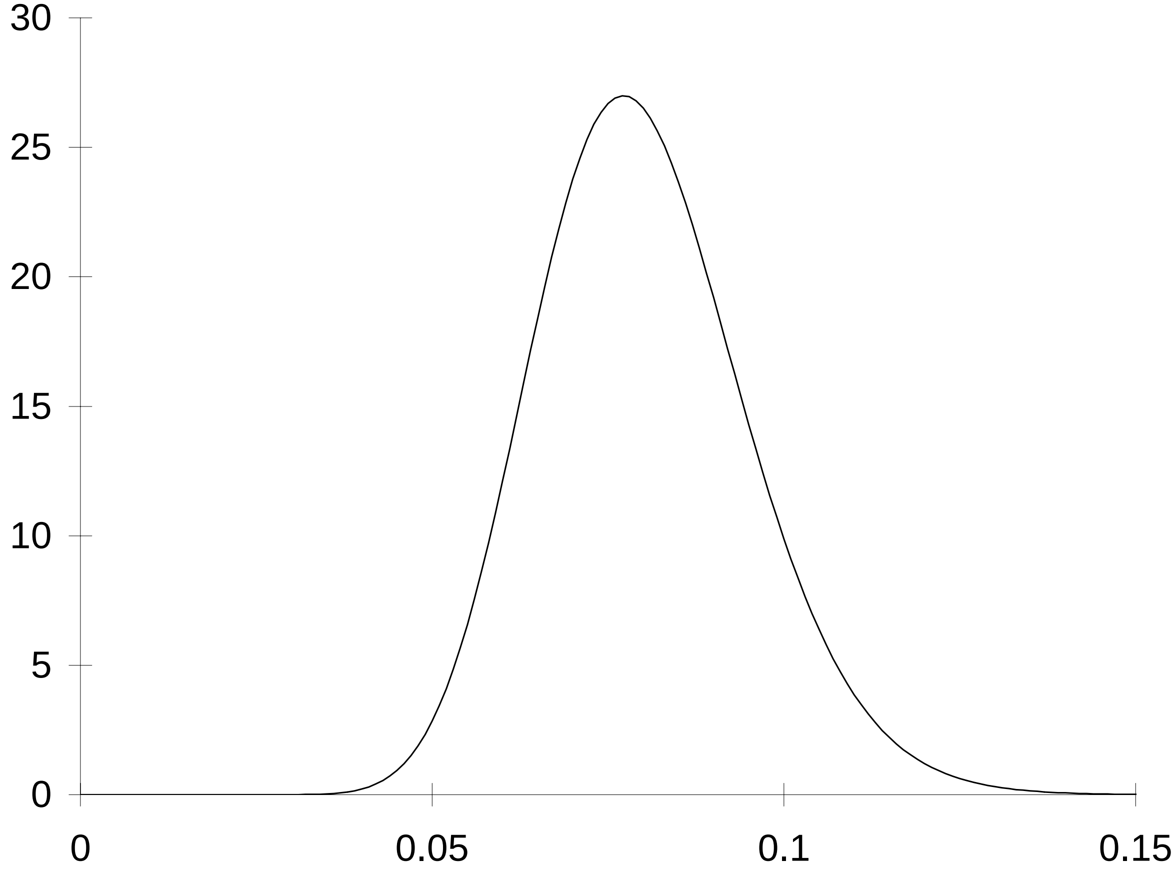
The CIR model takes the form

- $$dr = (\eta - \gamma r)dt + \sqrt{\alpha r}\, dX.$$

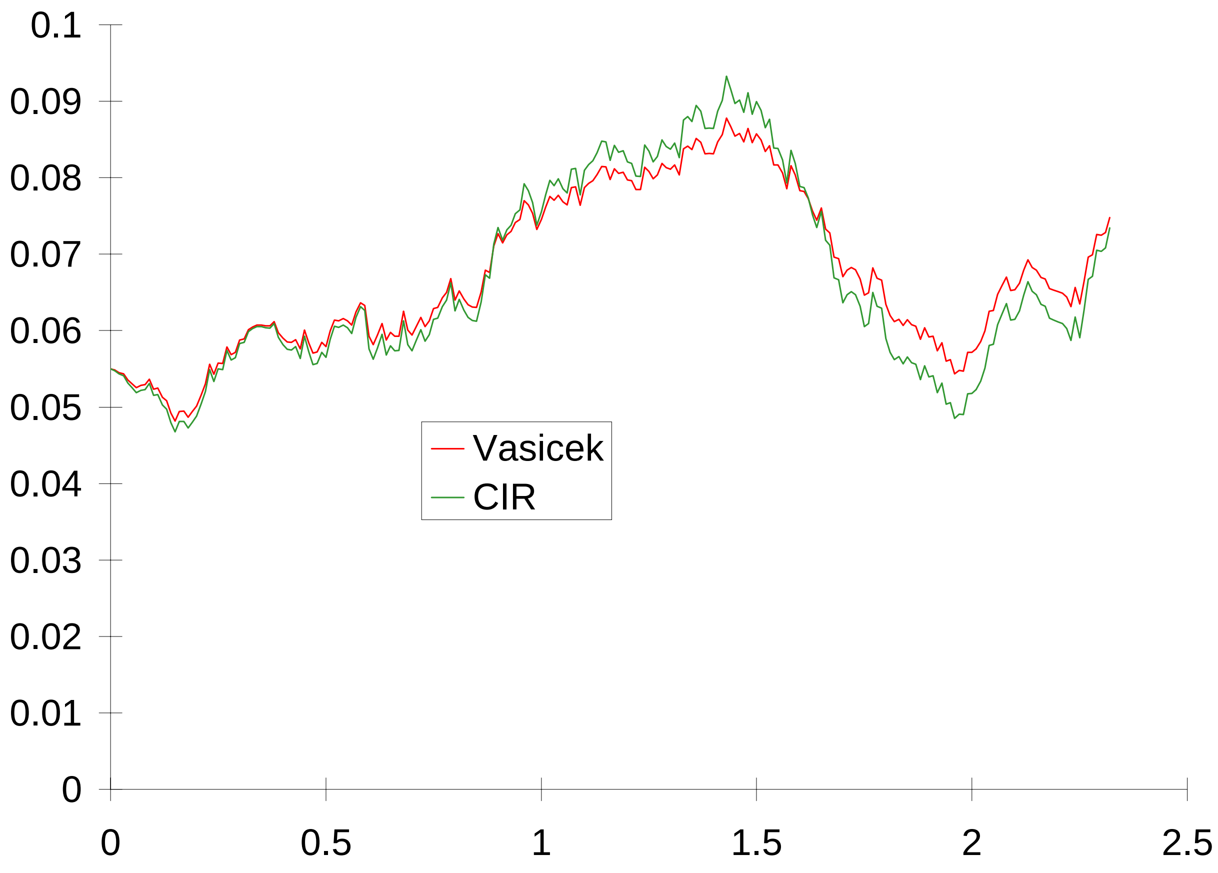
The spot rate is mean reverting and if $\eta > \alpha/2$ the spot rate stays positive. There are some explicit solutions for interest rate derivatives, although typically involving integrals of the non-central chi-squared distribution. The value of a zero-coupon bond is

$$e^{A(t;T) - rB(t;T)}$$

where A and B are as given above with $\beta = 0$. The resulting expression is not much simpler than in the non-zero β case.



The steady-state probability density function for the risk-neutral spot rate in the CIR model.



A simulation of the Vasicek and CIR models using the same random numbers.

Ho & Lee

Ho & Lee have

- $$dr = \eta(t)dt + \beta^{1/2}dX.$$

The value of zero-coupon bonds is given by

$$e^{A(t;T)-rB(t;T)}$$

where

$$B = T - t$$

and

$$A = -\int_t^T \eta(s)(T-s)ds + \frac{1}{6}\beta(T-t)^3.$$

This model was the first 'no-arbitrage model' of the term structure of interest rates. By this is meant that the careful choice of the function $\eta(t)$ will result in theoretical zero-coupon bonds prices, output by the model, which are the same as market prices.

- This technique is also called **yield curve fitting**.

This careful choice is

$$\eta(t) = -\frac{\partial^2}{\partial t^2} \log Z_M(t^*; t) + \beta(t - t^*)$$

where today is time $t = t^*$. In this $Z_M(t^*; T)$ is the market price today of zero-coupon bonds with maturity T.

Clearly this assumes that there are bonds of all maturities and that the prices are twice differentiable with respect to the maturity.

Hull & White

Hull & White have extended both the Vasicek and the CIR models to incorporate time-dependent parameters:

$$1)\quad dr = (\eta(t) - \gamma(t)r)dt + \beta(t)^{1/2}dX$$

$$2)\quad dr = (\eta(t) - \gamma(t)r)dt + \sqrt{\alpha(t)r}\,dX$$

This time dependence again allows the yield curve (and even a volatility structure) to be fitted.