1 Monte Carlo Methods For Pricing Options

The Monte Carlo Method

In this lecture...

- The justification for pricing by Monte Carlo simulation

- Implement the Monte Carlo method for simulating asset paths and pricing options

1.1 Introduction

More often than not we must solve option-pricing problems by numerical means.

It is rare to be able to find closed-form solutions for prices unless both the contract and the model are very simple.

The most useful numerical techniques are Monte Carlo simulations and finite-difference methods.

1.2 Relationship between derivative values and simulations

Simulations are at the very heart of finance.

With simulations we can explore the unknown future.

Simulations can also be used to price options.

The fair value of an option is the present value of the expected payoff at expiry under the risk-neutral random walk for the underlying. (Phelim Boyle 1977)

An amount of cash $M = M(t)$ in the bank grows according to

$$\frac{dM}{dt} = r(t) M$$

where $r(t)$ is the variable (risk-free interest rate). This differential equation has solution

$$M(t) = M(T) \exp\left(-\int_t^T r(\tau) \, d\tau\right)$$

i.e. the present value (time t) of a future cash flow (time T). The exponential term is the discount factor. In the simple case of a fixed rate of interest it becomes $e^{(-r(T-t))}$.

This gives the fair price of an option V to be

$$V = e^{(-r(T-t))} \mathbb{E}_{\mathbb{Q}} \left(\mathsf{Payoff}(S)\right)$$

where

$$\begin{aligned} S &= \text{asset price}, \;\; r = \text{interest rate}, \;\; T = \text{expiry}, \\ t &= \text{current time}, \mathbb{Q}\text{=risk neutral density}. \end{aligned}$$

Advantages

- Mathematics required is basic – complex path dependency can be incorporated with ease

- Models can be changed with little effort

- Correlations can be easily modelled,and it is easy to price options on many assets (high-dimensional contracts)

- There is plenty of software available, at the very least there are spreadsheet functions that will suffice for most of the time.

- To get better accuracy, simply perform more simulations.

Disadvantages

- Method slow in comparison to PDE's for low dimensional problems

- Difficult for early exercise (US Options)

- Calculating the Greeks

Scheme: The Monte Carlo method consists of the following steps

1. Simulate sample paths/realisations for the underlying asset price (e.g. equities or interest rates) over the relevant time horizon, according to the risk-neutral measure.

2. Evaluate the discounted cashflows of a derivative on each sample path, as determined by the structure of the security being priced.

3. Average the discounted cashflows over sample paths.

The important point to emphasize is that this is provided that the expectation is with respect to the risk-neutral random walk, not the real one ($\mu = r$).

So the option price becomes

$$e^{(-r(T-t))}.\frac{1}{N}\sum_{n=1}^{N}\text{Payoff}\,(S)$$

assuming that the interest rate is constant.

1.3 Numerical Scheme

The risk-neutral random walk for an asset price S is

$$dS = rSdt + \sigma SdX \tag{1.1}$$

Rewriting this continuous time process in discrete form

$$S_{i+1}^{(n)} = S_i^{(n)}\left(1 + r\delta t + \sigma\phi_i^{(n)}\sqrt{\delta t}\right) \tag{1.2}$$

gives a scheme for simulating n sample paths of the stock price $S(t)$.

In an earlier session, we saw an analytical solution of the risk-neutral lognormal random walk. That is particularly useful in the case of pricing a European option (path independent).

We now use the time-stepping form given by (1.2) from time t to T $(t \leq T)$.

The payoff for a European Call Option with strike E is $C = \max(S(T) - E,\ 0)$,

where $S(T)$ is obtained from (1.2) for each value $1 \leq n \leq N$.

Although we are not concerned with the path followed by the process $S(t)$ in getting to $S(T)$ we will nevertheless simulate this as we can price other options which are *path dependent*.

Based upon the N realisations an estimate for the price of an option becomes

$$\overline{C}\left(S,\ t\right) = \frac{1}{N}\sum_{n=1}^{N} C^{(n)}\left(S,\ T\right) \tag{1.3}$$

which is equivalent to

$$e^{(-r(T-t))}.\frac{1}{N}\sum_{n=1}^{N} \max\left(S^{(n)}\left(T\right) - E,\ 0\right) \tag{1.4}$$

1.4 Variance Reduction

For the simulation of an asset price in for example (2.1), samples are drawn from a probability (normal) distribution.

If these samples are generated in a fashion, which is not entirely random, but in a manner that reduces the fluctuations (i.e. volatility) of the resulting samples, computational time can be reduced considerably to obtain the desired degree of accuracy.

A similar effect can be obtained by performing suitable transformations on the function, which forms the basis of the simulation, so that dependency upon the fluctuations arising in the samples is reduced.

The disadvantage is that correlations are introduced.

It then becomes a choice, whether to compromise computational time over the risk of correlations being introduced.

Recall that the standard error ε associated with the Monte Carlo method is

$$\varepsilon = \frac{\sigma}{\sqrt{N}}.$$

Increasing the number of sample paths generated,by increasing N, leads to a reduction in ε. In addition we are able to manipulate the variance, i.e. reduce the value of σ. For this reason a whole area of mathematics exists, namely *variance reduction techniques.*

1.4.1 Antithetic Variable Technique

A powerful yet very simple technique by use of *antithetic variates*, which can reduce computational time and be implemented at no additional effort, was introduced to option pricing by Boyle (1977).

The method, which was initially used in the pricing of a European call option on a dividend paying stock, is outlined below.

The method is based upon the observation that if $\phi^{(n)} \sim N(0,1)$, then $-\phi^{(n)}$ also has a standard Normal distribution.

In this technique, by using the one set of random numbers generated, two estimates for an option are calculated. If ϕ_i is used to obtain $\overline{C}$, then $-\phi_i$ gives $\widehat{S}(t)$ and hence a second approximation for the option price $\widehat{C}$ where

$$\widehat{C} = e^{(-r(T-t))} . \frac{1}{N} \sum_{n=1}^{N} \max\left(\widehat{S}^{(n)}(T) - E,\ 0\right) \quad (2.5)$$

The estimate for the option C_μ is now the average of the two values, $\overline{C}$ & $\widehat{C}$, given by (2.4) and (2.5), so

$$C_\mu = \frac{\overline{C} + \widehat{C}}{2} \equiv \frac{1}{N}\sum_{n=1}^{N} \frac{\overline{C}^{(n)} + \widehat{C}^{(n)}}{2} \qquad (2.6)$$

The technique converges because of the symmetry of the Normal Distribution.

Justification for obtaining C_μ is based upon the distribution of the antithetic variates.

The pairs $\left\{\left(\phi^{(n)}, -\phi^{(n)}\right)\right\}$are distributed more regularly than a collection of $2n$ independent samples with the sample mean over the antithetic pairs always equal to the population mean of 0.

1.5 Other Options

1.5.1 Multi-Asset Options

In practice most options are written on several underlyings, called *Basket* or *Rainbow* options.

The lognormal random walk (GBM) given earlier for a single asset can be extended to consider the d dimensional case, where d is the number of underlying assets. Suppose that a number of d variables $S_1,\ S_2, ...,\ S_d$ are governed by the stochastic processes

$$\frac{dS_j}{S_j} = \mu_j dt + \sigma_j dX_j \quad (\ j = 1,\ .....,\ d) \tag{3.1}$$

where dX_j are increments of Wiener processes, again being random variables Normally distributed with mean and standard deviation 0 and $\sqrt{dt}$ respectively.

Hence $dX_j \sim N(0,\ dt)$ with $E\left(dX_j\right) = 0$ & $E\left(dX_j^2\right) = dt.$

In addition, the terms dX_l and dX_m are correlated such that , $E(dX_l dX_m) = \rho_{lm} dt$ with ρ_{lm} the correlation coefficient. μ_j & σ_j are the associated (real world) growth and volatility for each $\frac{dS_j}{S_j}$, in turn.

Consider an option , i.e. $V = V(S_1,\ S_2, ...,\ S_d,\ t)$ is a basket option with d underlyings. If V is a European call option with payoff $\max\left(S_j - E_j,\ 0\right)$, then the price of V for each $j = 1,\ ...,\ d$ using risk neutral valuation is given by

$$V = e^{(-r(T-t))} . \frac{1}{N} \sum_{n=1}^{N} \max\left(S_j^{(n)}(T) - E_j,\ 0\right)$$

where S_j is obtained from

$$S(T)^{(n)}\Big|_j = S(0)_j \exp\left\{\left(r - \tfrac{1}{2}\sigma_j^2\right) T + \sigma_j \phi_j^{(n)} \sqrt{T}\right\}$$

for each simulation $n = 1,\ ....,\ N$ and the Normally distributed random variables ϕ_i and ϕ_j are correlated.

1.5.2 Generating Correlated Normal Variables

Consider two uncorrelated standard Normal variables ε_1 and ε_2 from which we wish to form a correlated pair ϕ_1, & ϕ_2 $(\sim N(0,\ 1))$. The following scheme can be used

1. $E[\varepsilon_1] = E[\varepsilon_2] = 0$; $E\left[\varepsilon_1^2\right] = E\left[\varepsilon_2^2\right] = 1$ and $E[\varepsilon_1\varepsilon_2] = 0$ $(\because\ \varepsilon_1,\ \varepsilon_2$ are uncorrelated$)$.

2. Set $\phi_1 = \varepsilon_1$ and $\phi_2 = \alpha\varepsilon_1 + \beta\varepsilon_2$ (i.e. a linear combination).

3. Now $E\left[\phi_2^2\right] = \alpha^2 + \beta^2 = 1$ and $E[\phi_1\phi_2] = \alpha = \rho$ where ρ is a correlation coefficient.

4. This gives $\phi_1 = \varepsilon_1$ and $\phi_2 = \rho\varepsilon_1 + \left(\sqrt{1-\rho^2}\right)\varepsilon_2$ which are correlated standardised Normal variables.

1.5.3 Path Dependency

These represent an exciting class of options for investors and consist of derivatives with more complicated payoff conditions than the plain vanilla options discussed earlier.

Barrier options are similar to standard options except that they are distinguished or activated when the underlying asset price reaches a predetermined *barrier.* The payoff of a standard European style option is based on the price on the underlying asset on the expiration date. These are path-independent since it does not matter what path the underlying asset followed during the life of the option.

Barrier options are path-dependent since they are dependent on the price movement of the underlying asset. A knock-out option will expire early if the barrier price is reached whereas a knock-in option will come into existence if the barrier is reached.

An option whose payoff depends on the average price of the underlying asset over the time horizon is called an *Asian Option*. These are sometimes referred to as *Average options.* For a prospective investor these type of option contracts are attractive because they tend to cost less than regular options. The reason for this is that the averaging process tends to dampen the volatility effects.

They are commonly traded on currencies and commodity products which have low trading volumes. They were originally used in 1987 when Banker's Trust Tokyo office used them for pricing average options on crude oil contracts; and hence the name "Asian" option.

The cost of an Asian Call and Put option is given in turn by

$$
\begin{aligned}
V_{\text{call}} &= e^{(-r(T-t))}.\frac{1}{N}\sum_{n=1}^{N}\text{max}\left(\widetilde{S}^{(n)} - E,\ 0\right) \\
V_{\text{put}} &= e^{(-r(T-t))}.\frac{1}{N}\sum_{n=1}^{N}\text{max}\left(E - \widetilde{S}^{(n)},\ 0\right)
\end{aligned}
$$

where the sampling

$$\widetilde{S} = \frac{1}{N}\sum_{j=1}^{N} S_j$$

for each t_j over the total time range and $n = 1, \ ..., \ N$ simulations. Each value of the asset price $S(t)$ is required for evaluating an Asian option, where as in the previous cases the final value $S(T)$ is only used, therefore

$$S(T)^{(n)} = S(0) \exp\left\{\left(r - \tfrac{1}{2}\sigma^2\right) T + \sigma\phi^{(n)}\sqrt{T}\right\}$$

cannot be used, but instead (2.2) to generate each $S(t)$.

For each sample path the values of $S(t)$ are averaged to give $\widetilde{S}$ and the payoff calculated. The payoff is averaged to yield a final value which is then discounted using the risk-free interest rate to obtain the present value.

Lookback Options are another type of exotic option (which may be priced using the sample paths for the asset prices generated). Their payoff depends upon an observed maximum or minimum of the underlying over some future period prior to expiry. This type of option is generally very costly. The payoff from the lookback call is the terminal price of the underlying less the minimum value

$$C_T = e^{(-r(T-t))} . \frac{1}{N} \sum_{n=1}^{N} \max \left(S^{(n)} \Big|_{\max} - S_T,\ 0 \right)$$

where

$$M = \begin{cases} \max\limits_{\tau \in [t,T]} S_\tau \\ \min\limits_{\tau \in [t,T]} S_\tau \end{cases}$$

is the observed maximum or minimum.

Another type of option available is with a strike E and payoff

$$\max (M - E,\ 0) .$$

1.6 Modelling Stochastic Interest Rates

Pricing options with a constant interest rate provides a rather simplified and ideal setting. In reality this is done with stochastic interest rates. Hence the pricing scheme becomes

$$\mathbb{E}_{\mathbb{Q}}\left(\left[\text{exp} - \int_t^T r(\tau)\, d\tau\right] \text{Payoff}(r)\right)$$

This type of modelling introduces additional difficulty as we do not know what the value of $r(t)$ is. Consider the Vasicek model for the short rate given by

$$dr = (\eta - \gamma r)\, dt + \sigma dX.$$

This can be written as

$$dr = \gamma(\overline{r} - r)\, dt + \sigma dX, \quad \text{with} \quad \overline{r} = \eta/\gamma$$

where γ is the reversion rate, $\overline{r}$ the mean rate and σ volatility.

The algorithm for pricing options in a stochastic interest rate environment is as follows:

1. Simulate the random walk for the risk-adjusted spot interest rate r, over the relevant time horizon T. This time period commences with today's value of the spot rate, until the expiry of the option. This gives one sample path for the spot rate.

2. For such a realisation calculate the average spot-rate r_{aver} over the time horizon. At expiry obtain the payoff $P(r_T) = \max(r_{\text{aver}} - r_{\text{strike}},\ 0)$.

3. Perform many such realisations by repeating the first two steps.

4. For each realisation calculate the present value of the payoff by discounting at the average rate, i.e. PV$= P\left(r_T\right)\exp\left(-r_{\text{aver}}T\right).$

5. Calculate the average of all the PV terms obtained, this is the option value.

We end by looking at how to price options using stochastic interest rates. Consider the problem of pricing an equity option which evolves according to the lognormal random walk and the Vasicek model is used to obtain the spot-rate

$$dS = rSdt + \sigma Sd\widetilde{X}$$

$$dr = \gamma\left(\overline{r} - r\right)dt + \sigma d\widehat{X}$$

where

$$d\widetilde{X} = \varepsilon_1\sqrt{dt}, \quad d\widehat{X} = \varepsilon_2\sqrt{dt}$$

such that

$$\mathbb{E}\left[d\widetilde{X}d\widehat{X}\right] = \rho dt.$$

Using the earlier result for modelling correlations the problem becomes

$$dS = rSdt + \sigma S\varepsilon_1\sqrt{dt}$$

$$dr = \gamma\left(\overline{r} - r\right)dt + \sigma\left(\rho\varepsilon_1 + \left(\sqrt{1-\rho^2}\right)\varepsilon_2\right)\sqrt{dt}$$

where $\varepsilon_1,\ \varepsilon_2$ are uncorrelated standard normal variables.

Call today $t = 0$, so our assets are S_0 and r_0 which are substituted into

$$dS = rSdt + \sigma Sd\widetilde{X}, \qquad dr = \gamma\left(\overline{r} - r\right)dt + \sigma d\widehat{X}$$

and perform the following computation

$$\begin{aligned} S_1 &= S_0 + dS = S_0 + \left(r_0 S_0 dt + \sigma S_0 d\widetilde{X}\right) \\ r_1 &= r_0 + dr = r_0 + \left(\gamma\left(\overline{r} - r_0\right)dt + \sigma d\widehat{X}\right) \end{aligned}$$

then

$$\begin{aligned} S_2 &= S_1 + dS = S_1 + \left(r_1 S_1 dt + \sigma S_1 d\widetilde{X}\right) \\ r_2 &= r_1 + dr = r_1 + \left(\gamma\left(\overline{r} - r_1\right)dt + \sigma d\widehat{X}\right) \end{aligned}$$

and so on, till expiry T is reached. At $t = T$ the payoff is calculated and discounted using $\exp\left[-\int_t^T r(\tau)\,d\tau\right]$.

How do we obtain $\int_t^T r(\tau)\,d\tau$?

We have values of the spot rate r for each τ over the interval $[0, T]$, so we can use a simple numerical integration technique such as the Trapezoidal rule

$$\int_0^T r(\tau)\, d\tau = \frac{h}{2}\left[r_0 + 2\sum_{i=1}^{M-1} r_i + r_M\right]$$

where, for M steps, the fixed step-length $h = T/M$ and $r_i = ih$.

This is one realisation!

We now repeat the above several times and take the average of all these sample paths.

This gives the option price.

Vasciek is an *equilibrium* interest rate model.

Cox-Ingersoll-Ross interest rate model (aka "Square Root Process")

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r} dX$$

Cox-Ingersoll-Ross is an *equilibrium* interest rates model.

Ho-Lee interest rate model

$$dr = \eta(t)dt + \sqrt{\sigma} dX$$

Ho-Lee is a *no-arbitrage* interest rates model.

Hull-White interest rate models

$$dr = (\eta(t) - \gamma(t)r)dt + \beta(t)^{1/2} dX$$

$$dr = (\eta(t) - \gamma(t)r)dt + \sqrt{\alpha(t)r} dX$$

Hull-White are *no-arbitrage* interest rates models.

Multi-factor interest rates models

Brennan and Schwartz

$$dr = (a_1 + b_1(l - r))dt + \sigma_1 r dX_1$$

$$dl = (a_2 + b_2(l - r))dt + \sigma_2 l dX_2$$

Fong and Vasicek

$$dr = a(\bar{r} - r)dt + \sqrt{\xi} dX_1$$

$$d\xi = b(\bar{\xi} - \xi)dt + c\sqrt{\xi} r dX_2$$

Longstaff and Schwartz

$$dx = a(\bar{x} - x)dt + \sqrt{x} dX_1$$

$$dy = a(\bar{y} - y)dt + \sqrt{y} dX_2$$

Popular stochastic volatility models

GARCH-diffusion

$$dv = (a - bv)dt + cv dX$$

Square-Root model (Heston)

$$dv = (a - bv)dt + c\sqrt{v}dX$$

(popular because it has a "closed form solution" for European options)

3/2 model

$$dv = (av - bv^2)dt + cv^{3/2}dX$$

(popular because it has a "closed form solution" for European options)

Ornstein-Uhlenbeck process

$$dy = (a - by)dt + cdX$$

where $y = \ln v$ (popular because it matches data well)

Hull-White model

$$d\sigma^2 = a(b - \sigma^2)dt + c\sigma^2 dX$$

2 The Greeks

Having discussed the pricing of options, we now look at hedge parameters known as the greeks. It is a fairly simple task of defining *theta*, *delta* and *gamma*:

$$\theta_n^m \sim \frac{V_n^m - V_n^{m-1}}{\delta t}, \quad \Delta_n^m \sim \frac{V_{n+1}^m - V_{n-1}^m}{2\delta S},$$
$$\Gamma_n^m \sim \frac{V_{n-1}^m - 2V_n^m + V_{n+1}^m}{\delta S^2}.$$

These are the simple greeks which involve derivatives with respect to a variable. For more advanced greeks (based upon parameters), such as *vega* and *rho,* these can be calculated using FDM by expressing modifications of the BSE. To illustrate this idea let us introduce a convenient form of shorthand by defining the options *vega* υ as

$$\upsilon\left(S,t\right) = \frac{\partial V}{\partial \sigma}.$$

Now by differentiating the Black-Scholes problem (equation and payoff)

$$\frac{\partial}{\partial\sigma}\left\{\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+(r-D)\,S\frac{\partial V}{\partial S}-rV=0\right\}$$

and

$$\frac{\partial}{\partial\sigma}\left\{\max\left(S-E,\ 0\right)\right\}$$

the equation becomes

$$\left\{\frac{\partial}{\partial t}\frac{\partial V}{\partial\sigma}+\frac{1}{2}S^2\frac{\partial}{\partial\sigma}\left(\sigma^2\frac{\partial^2 V}{\partial S^2}\right)+(r-D)\,S\frac{\partial}{\partial S}\frac{\partial V}{\partial\sigma}-r\frac{\partial V}{\partial\sigma}=0\right\}$$

to give the final form

$$\frac{\partial v}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 v}{\partial S^2}+(r-D)\,S\frac{\partial v}{\partial S}-rv=-\sigma S^2\frac{\partial^2 V}{\partial S^2}$$

together with the payoff condition which becomes

$$\frac{\partial V\left(S,T\right)}{\partial\sigma}=v\left(S,t\right)=0.$$

The resulting PDE is the BSE with a forcing term (on the right hand side) due to the diffusion. Note - in order to obtain this we have assumed the existence of continuous second order partial derivatives so for example

$$\frac{\partial}{\partial \sigma}\frac{\partial V}{\partial t} \equiv \frac{\partial}{\partial t}\frac{\partial V}{\partial \sigma} \rightarrow \frac{\partial \upsilon}{\partial t}.$$

We also note that in arriving at $\upsilon(S,t) = 0$ we have assumed that the payoff is independent of the volatility.