Numerical PDEs
in
Option Pricing Models

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Outline

1. Assets in Financial Markets

   - Statistics of asset price motion
   - Derivative securities

2. Black--Scholes hedging

   - Elimination of risk
   - The link between SDEs and PDEs
   - The Black--Scholes PDE

3. Numerical Methods for advanced cases

   - American options and free boundaries
   - Exotic options
   - Multi--asset American options

4. Future work and challenges

   - Jump diffusion models
   - Models with transaction costs
   - Multiscale stochastic volatility
   - Stochastic optimal control models

1.Assets in Financial Markets

- Stocks

- Stocks Indices:
  Dow Indust, S&P500, NASDAQ

- Interest rates
  Different durations, different credit quality

- Commodities:
  pork bellies, wheat, natural gas

- Precious metals:
  gold, silver

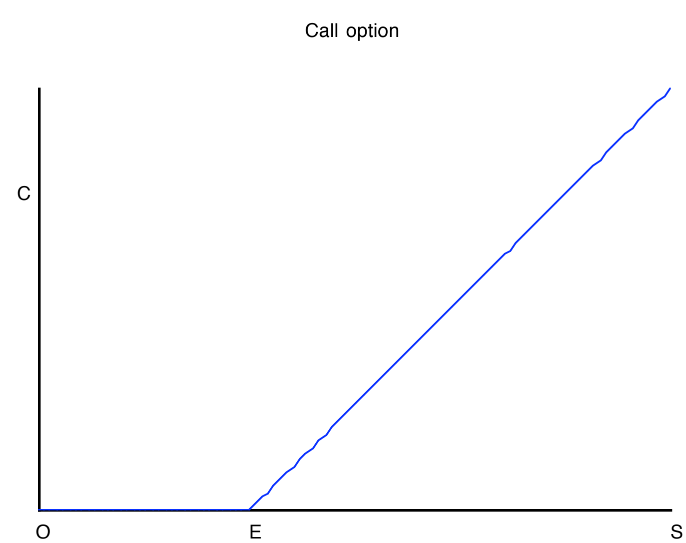
Theory: fair price = **E**(discounted future earnings)
Reality: move ``randomly in response to natural & economic events, and popular expectations

Derivative security

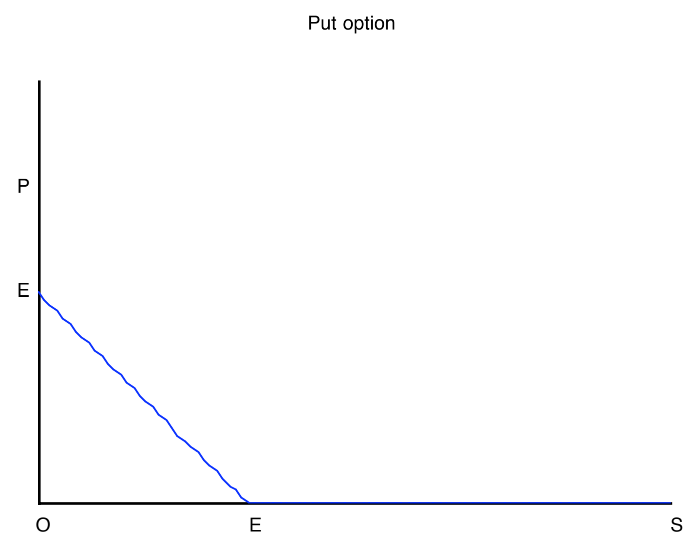
Value depends on price of some other asset or assets

- *Forward (futures) contract*
  Commitment (no choice) to purchase asset S for price E at date T in the future. (Long or short side)

- *Call option*
  Have *option* to buy asset S for price E at date T in the future. (Person who buys makes the choice)



- *Put option*
  Have *option* to sell asset S for price E at date T in the future. (Person who sells makes the choice)



1. Options are extremely attractive to investors, both for *speculation* and *hedging*.

2. There is a systematic way to determine how much options are worth, and so they can be bought and sold with some confidence.

Examples

- Stock options for speculation
(large possible gain with small initial investment)

- S&P500 put options to protect portfolio value

- Airlines and utilities lock in fuel price

- Farmers sell crop before they know how much

- Mortgage prepayment

Ways to exchange *risk* between willing participants

- **European option**
  Option to buy or sell can be exercised *only* at date T
  Passive--wait and see what you get

- **American option**
  Option to buy or sell can be exercised once *at anytime up to and including* T
  Need to make *exercise decision* at each time up to T

``Vanilla call and put options traded on *exchanges*.

``Exotic custom options traded *over the counter* between large financial institutions

Use mathematics to determine value of derivative in terms of statistics of motion of *underlying* asset

How is option price determined?

- **Intuition**
  (Louis Bachelier, *Théorie de la Spéculation*, Paris 1900)
  Value of option = Expected value of pay-off
  Maybe add some penalty for riskiness
  Compute expectation: Fokker--Planck PDE for value

- **Dynamic Hedging**
  Black \ Scholes \ Merton 1973 (Nobel Prize 1997)
  Eliminate option risk by trading underlying asset
  Unique ``risk--free value V for option
  ⋆ Establishment of derivatives exchanges. The first of these, the Chicago Board Options Exchange (CBOE), started in 1973 and there are more than 50 throughout the world in 2004
  ⋆ Foundation of ``modernfinance

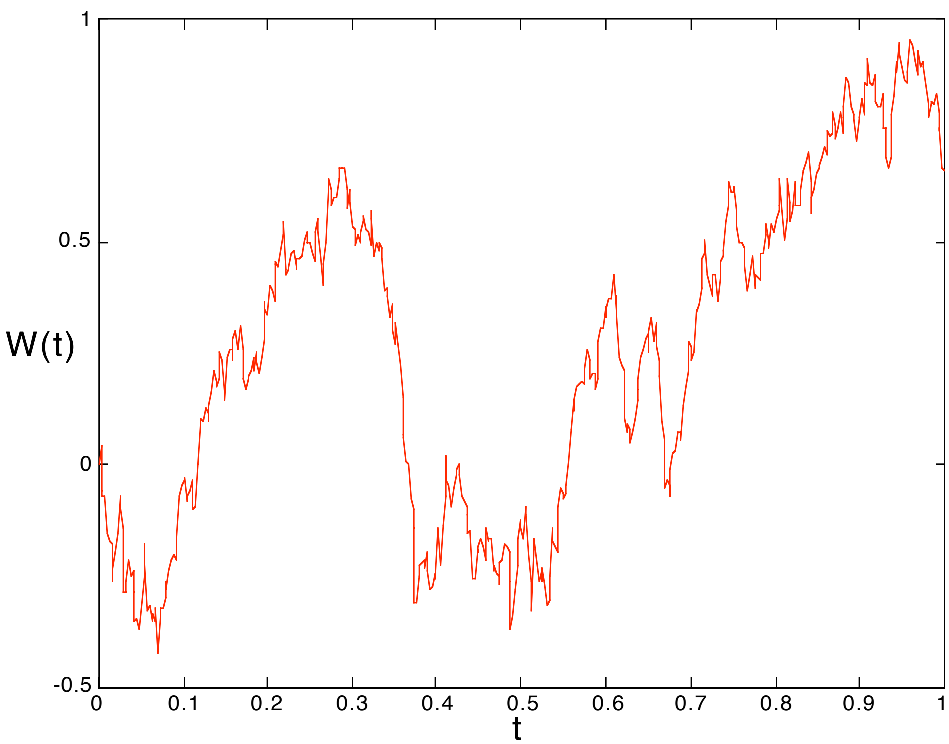
SDE

Stochastic Differential Equation (SDE) models for asset prices $S(t)$:

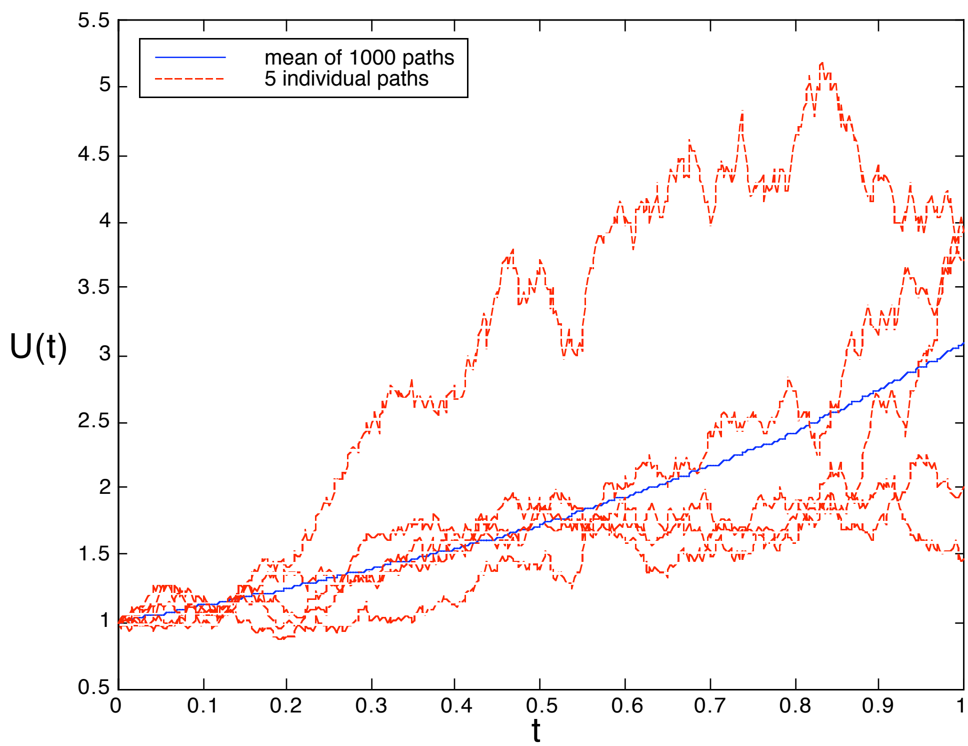
$$dS(t) = a(S,t)dt + b(S,t)dW(t)$$

$W(t)$ is a Brownian motion (Wiener process)

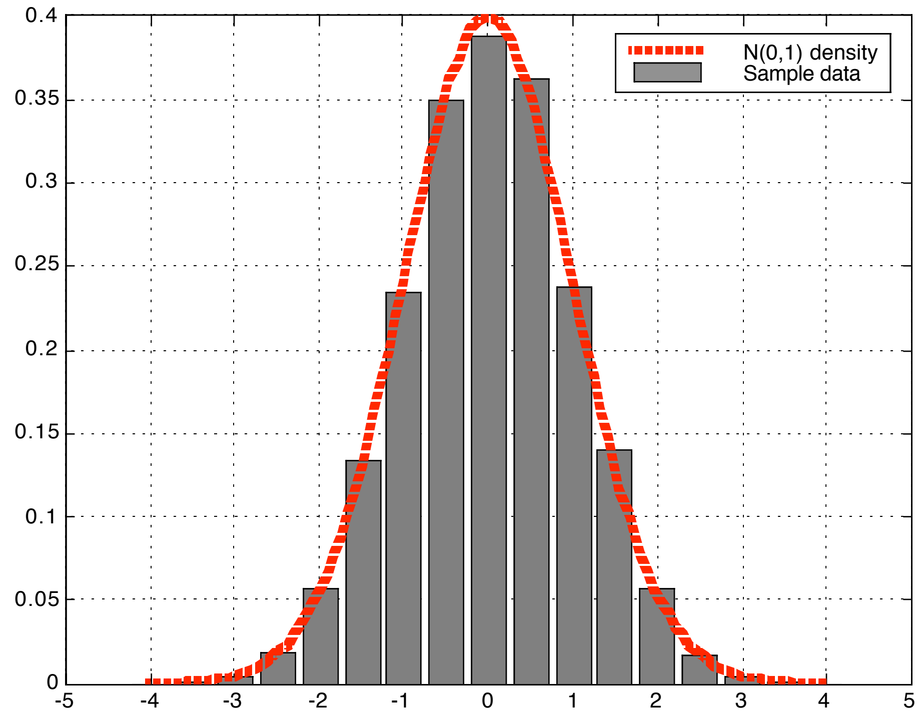
The *standard model* for stock prices has $a(S,t) = \mu S$ and $b(S,t) = \sigma S$



Discretized Brownian path



The function $U(W(t))$, solution of a linear SDE, averaged over 1000 discretized Brownian paths and along 5 individual paths



Density estimate for sample data generated randomly (Central Limit Theorem)

Higham (2004) Cambridge University Press

2. Black Schole Hedging

Link between SDEs and PDEs
Ito s Lemma

$$dF(S,t) = \left(a\frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{b^2}{2}\frac{\partial^2 F}{\partial S^2} \right) dt + b\frac{\partial F}{\partial S} dW(t)$$

- $S(t)$ follows a Wiener process
- $\frac{\partial F}{\partial S}, \frac{\partial^2 F}{\partial S^2}, \frac{\partial F}{\partial t}$, are continuous

asset with value $S(t)$:

$$dS(t) = a(S,t)dt + b(S,t)dW(t)$$

riskless bond with value $B(t)$:

$$dB(t) = rB(t)dt$$

Assumptions

1. asset and bond can be bought and sold without restriction (bond provides a riskless return)

2. no arbitrage (any portfolio with a riskless return satisfies same equation as $B(t)$)

$V(t)$: value of a European--style contract
$\Pi(t)$: value of a portfolio

$$\Pi(t) = V(t) - \Delta S(t)$$

$$d\Pi(t) = dV(t) - \Delta dS(t)$$

Ito s Lemma

$$\Downarrow$$

$$dV(t) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS(t) + \frac{1}{2}b(S(t),t)^2\frac{\partial^2 V}{\partial S^2})dt$$

$$\Downarrow$$

$$d\Pi(t) = \left(\frac{\partial V}{\partial S} - \Delta\right)dS(t) + \left(\frac{\partial V}{\partial t} + \frac{1}{2}b(S(t),t)^2\frac{\partial^2 V}{\partial S^2}\right)dt$$

eliminate uncertainty

$$\Rightarrow \Delta = \frac{\partial V}{\partial S}$$

$$d\Pi(t) = \left(\frac{\partial V}{\partial t} + \frac{1}{2} b(S(t),t)^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

no--arbitrage

$$\Downarrow$$

$$d\Pi(t) = r\Pi(t)dt$$

$$\Downarrow$$

$$\frac{\partial V}{\partial t} + \frac{1}{2} b(S(t),t)^2 \frac{\partial^2 V}{\partial S^2} = r\Pi = rV - rS\frac{\partial V}{\partial S}$$

when $b(S,t) = \sigma S$

$$\frac{\partial V}{\partial t} - rV + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \quad = \quad 0$$

Black--Scholes Model

$$\frac{\partial V}{\partial t} - rV + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \;=\; 0$$

In 1997, the Nobel Prize in Economics was awarded for the work that led to Black--Scholes option pricing theory. The theory provides a sophisticated way of analyzing and understanding the relationships among stock options, option prices and expected stock market volatility.

The goal of the theory of option pricing is to be able to compute *a fair* price for an option -- it is not to estimate what the price of the underlying asset might be in the future.

Black and Scholes, The pricing of options and corporate liabilities, J. Pol. Econ. 81, pp. 637--659, 1973.

Merton, Theory of rational option pricing, Bell I. Econ. Manag. Sci. 4, pp. 141--183, 1973.

Black--Scholes Model
(European call option)

$$\frac{\partial C}{\partial t} - rC + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \;=\; 0$$

$C(S, t)$-- expected value of the option

S-- asset price

$r(t)$-- rate from alternative riskless investments

$\sigma(t)$-- asset volatility

Final and Boundary Conditions for call options

At the future time T when the option is exercised, the final call value is

$$C(S,T) \;=\; \max(S-E,0) for S \geq 0,$$

where

S--actual share price
E--strike price

The boundary conditions are

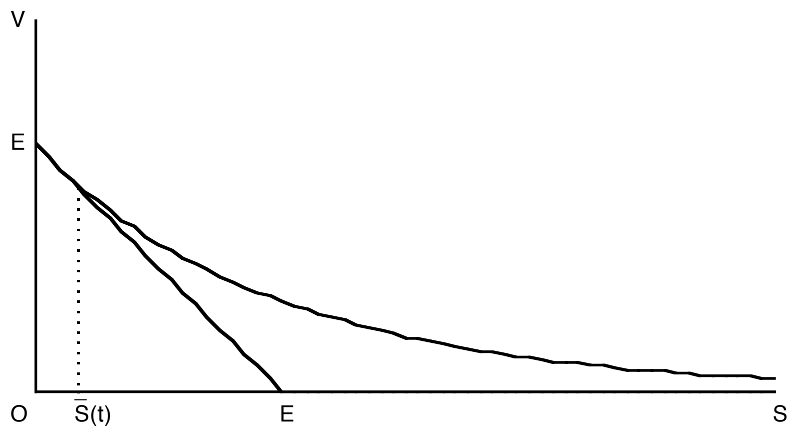
$$C(0,t) = 0 \& C(S,t) \approx S, S \to \infty$$

3. Numerical methods for advanced cases

American put option

$$
\begin{aligned}
\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} &= rP, \\
S > \overline{S}(t), \qquad & 0 \le t < T \\
P(S,T) &= \max(E - S, 0),\ S \ge 0, \\
\frac{\partial P}{\partial S}(\overline{S}, t) &= -1, \\
P(\overline{S}(t), t) &= E - \overline{S}(t), \\
\lim_{S\to\infty} P(S,t) &= 0, \\
\overline{S}(T) &= E, \\
P(S,t) &= E - S, 0 \le S < \overline{S}(t).
\end{aligned}
$$

American put option



$$P(S,t) \;\geq\; \max(E-S,0), S \geq 0, 0 \leq t \leq T$$

Penalty Term

$0 < \epsilon << 1$ a small regularization parameter

$$\begin{aligned} \frac{\partial V_\epsilon}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_\epsilon}{\partial S^2} + rS\frac{\partial V_\epsilon}{\partial S} - rV_\epsilon & & S \in [0, S_\infty] \\ + \frac{\epsilon C}{V_\epsilon + \epsilon - q(S)} &= 0, \end{aligned}$$

$$\begin{aligned} V_\epsilon(S,T) = \max(E - S, 0), \\ V_\epsilon(0,t) = E, \\ V_\epsilon(S_\infty, t) = 0, \end{aligned}$$

$$C \geq rE, q(S) = E - S$$

Numerical Methods

- **Objective**: Accuracy at Minimum Cost (not at any cost)

- **Numerical Accuracy**: Error Analysis

- **Numerical Stability**: Stability Analysis

- **Numerical Efficiency**: Minimum Cost

- **Reliability & Flexibility**: Reduce Preparation and Debugging Time

- **Visualization**: Graphs and Convergence Tables

Discretization & Roundoff Errors

$$f'_{comp}(c) = \frac{f(c+h) + \epsilon_+ - f(c-h) - \epsilon_-}{2h}$$
$$= \frac{f(c+h) - f(c-h)}{2h} + \frac{\epsilon_+ - \epsilon_-}{2h}.$$

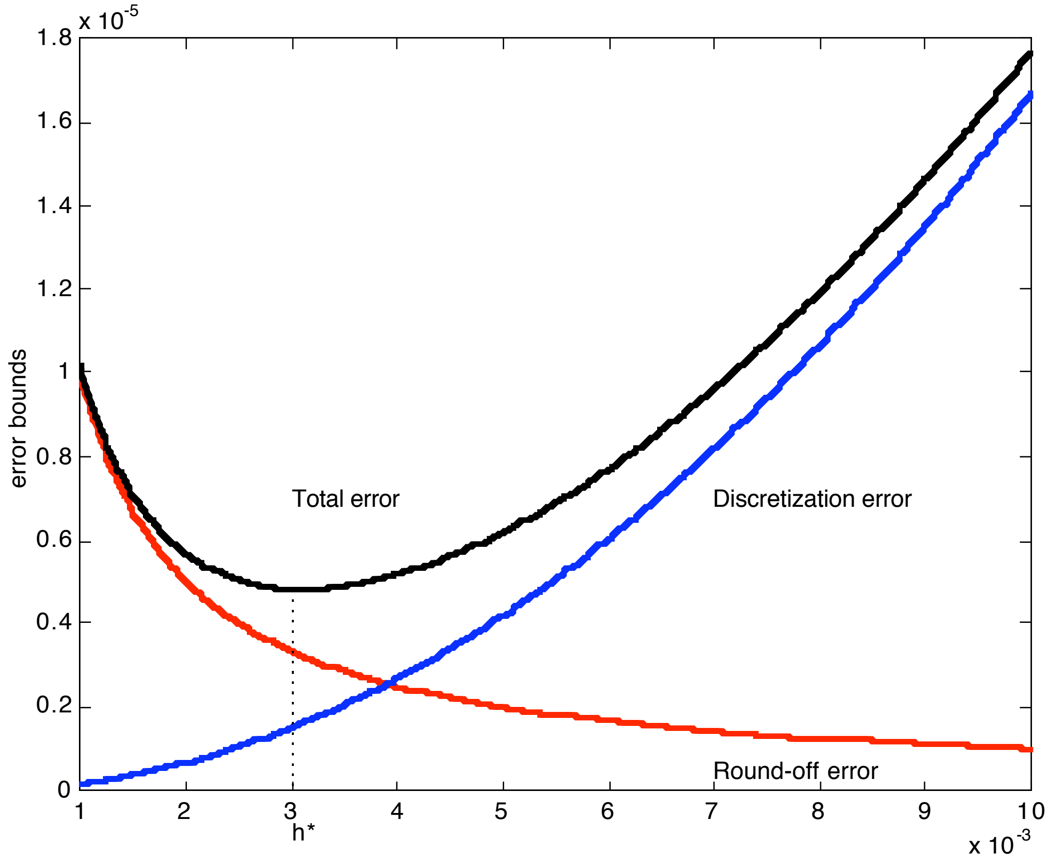
$$f'(c) = f'_{comp}(c) - \frac{1}{6}h^2 f'''(\xi) - \frac{\epsilon_+ - \epsilon_-}{2h},$$

$$|\epsilon_+| < \epsilon, |\epsilon_-| < \epsilon, |f'''(x)| \leq M$$

$$\Downarrow$$

$$|f'(c) - f'_{comp}(c)| \leq \frac{1}{6}Mh^2 + \frac{\epsilon}{h}$$

Optimal Value of h: $h^* = \sqrt[3]{\frac{3\epsilon}{M}}$



Errors in the central difference approximation

Fully--Discrete Approach
θ--methods

spatial derivatives $\rightarrow$ central differencing

$$\frac{V_j^{n+1}-V_j^n}{\Delta t}+\frac{1}{2}\sigma^2 S_j^2[\theta\frac{\delta_S^2 V_j^{n+1}}{\Delta S^2}+(1-\theta)\frac{\delta_S^2 V_j^n}{\Delta S^2}]$$
$$+rS_j[\theta\frac{\Delta_S V_j^{n+1}}{2\Delta S}+(1-\theta)\frac{\Delta_S V_j^n}{2\Delta S}]$$
$$-r[\theta V_j^{n+1}+(1-\theta)V_j^n]$$
$$+\theta\frac{\epsilon C}{V_j^{n+1}+\epsilon-q(S_j)}+(1-\theta)\frac{\epsilon C}{V_j^n+\epsilon-q(S_j)} \quad = \quad 0$$

$$\delta_S^2 V_j^n = V_{j+1}^n - 2V_j^n + V_{j-1}^n$$

$$\Delta_S V_j^n = V_{j+1}^n - V_{j-1}^n$$

nonlinear penalty term

$\downarrow$

nonlinear system of equations

Well-known θ--methods

- Forward Euler ($\theta = 1$)
- Crank--Nicolson ($\theta = \frac{1}{2}$)
- Backward Euler ($\theta = 0$)

Linearly implicit θ--methods

penalty term treated explicitly

$\downarrow$

linear system of equations

$$V_j^n \to V_j^{n+1}$$

$$\begin{aligned}\frac{V_j^{n+1}-V_j^n}{\Delta t} + \frac{1}{2}\sigma^2 S_j^2[\theta\frac{\delta_S^2 V_j^{n+1}}{\Delta S^2} + (1-\theta)\frac{\delta_S^2 V_j^n}{\Delta S^2}] & \\ +rS_j[\theta\frac{\Delta_S V_j^{n+1}}{2\Delta S} + (1-\theta)\frac{\Delta_S V_j^n}{2\Delta S}] & \\ -r[\theta V_j^{n+1} + (1-\theta)V_j^n] & \\ +\frac{\epsilon C}{V_j^{n+1}+\epsilon - q(S_j)} & = 0,\end{aligned}$$

$$V(S,t) \geq \max(E-S,0), S \geq 0, 0 \leq t \leq T$$

Crank Nicolson method: incurs spurious oscillations

Pooley, *et al.*, 2003

Voss and Khaliq, 2003

Heston and Zhou, 2000

Zvan, *et al.*, 1999

Conventional form $\rightarrow \eta = T - t$

$$\frac{\partial V}{\partial \eta} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV + \frac{\epsilon C}{V + \epsilon - q(S)},$$
$$S \in [0, S_\infty], \eta \in (0, T],$$

$$V(S, \eta = 0) = \begin{cases} \max(S - E, 0) & \text{for a call} \\ \max(E - S, 0) & \text{for a put} \end{cases}$$

$$V(0, \eta) \rightarrow V_\eta = -rV + \frac{\epsilon C}{V + \epsilon - E}$$

$$V(S = \infty, \eta) \rightarrow \begin{cases} \approx S & \text{for a call} \\ \approx 0 & \text{for a put} \end{cases}.$$

Semi--Discrete Approach
Method Of Lines (MOL)

$$\Delta S = \frac{S_\infty}{M+1}$$

nonlinear system of ODEs

$$\frac{d\mathbf{v}}{d\eta} = \mathbf{F}(\eta, \mathbf{v}), \mathbf{v}(0) = \mathbf{v}_0,$$

$$\mathbf{F}(\eta, \mathbf{v}) = A\mathbf{v} + \mathbf{g}(\eta, \mathbf{v})$$

$A \rightarrow M \times M$ nonsymmetric tridiagonal (central difference approximations to the spatial derivatives)

$\mathbf{g}(\eta, \mathbf{v}) \rightarrow$ nonlinear discretized penalty term

second order Adams--Moulton formula

$$\mathbf{v}^{n+1} - \mathbf{v}^n \ = \ k[\frac{1}{2}\mathbf{F}^n + \frac{1}{2}\mathbf{F}^{n+1}]$$

split form

$$\mathbf{v}^{n+1} - \mathbf{v}^n \ = \ k[\frac{1}{2}\mathbf{F}^n + (\frac{1}{2} - \phi)\mathbf{F}^{n+1}] + \phi k \mathbf{F}^{n+1}$$

predicted value of the function, $\mathbf{F}^{n+1}$, is obtained from a separate *implicit* predictor

$$\mathbf{v}^{n+1} - \mathbf{v}^n \ = \ k[(\frac{1}{2} + \phi)\mathbf{F}^n + (\frac{1}{2} - \phi)\mathbf{F}^{n+1}],$$

$\phi \neq 0, \frac{1}{2}$.time step $k = \Delta\eta$

BDFs: Cash (1983)
AMFs: Voss & Casper (1989)

linearly Implicit Schemes

Treat the nonlinear term $\mathbf{g}(\eta, \mathbf{v})$, explicitly, in both the predictor and corrector through evaluation of $\mathbf{g}$ at the most current available values

tridiagonal coefficient matrix

$$T = [I - k(\frac{1}{2} - \phi)A]$$

$$\begin{aligned} T\mathbf{v}^{n+1} &= [I + k(\frac{1}{2} + \phi)A]\mathbf{v}^n + k\mathbf{g}^n \\ T\mathbf{v}^{n+1} &= [I + \frac{k}{2}A]\mathbf{v}^n + \frac{k}{2}[\mathbf{g}^n + \mathbf{g}^{n+1}] + \phi k A \mathbf{v}^{n+1} \end{aligned}$$

accurate $O(\Delta\eta^2, \Delta S^2)$

Condition number & Stiffness ratio

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ \mathbf{r} &= \mathbf{b} - A\hat{\mathbf{x}} \end{aligned}$$

$\hat{\mathbf{x}}$, computed solution
Relationship between the size of the relative error and that of the relative residual

$$\frac{1}{cond(A)}\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \leq \frac{\|\mathbf{x} - \hat{\mathbf{x}}\|}{\|\mathbf{x}\|} \leq cond(A)\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|}$$

If $cond(A) >> 1$, a small relative residual does not guarantee a small relative error

$$cond(A) = \|A\|_2\|A^{-1}\|_2 = \rho(A)\rho(A^{-1}), A^T = A$$

$$\begin{aligned} A_{N\times N} &= trid(1, -2, 1) \\ &\Rightarrow cond(A) = O(h^{-2}) = O(N^2) \end{aligned}$$

$$cond(A^k) = O(N^{2k})$$

$$cond(A) >> 1$$

$$\downarrow$$

practical effect

$$\nearrow \nwarrow$$

data accuracy machine word length

$$cond(A) = 10^6$$

$$\Downarrow$$

possible loss of 6 decimal digits

(disastrous effects on a computer with word length $\equiv$ 8 decimal digits)

Stability & Positivity

linear stability properties

$$\frac{dv}{dt} = \lambda v,\ Re(\lambda) < 0$$

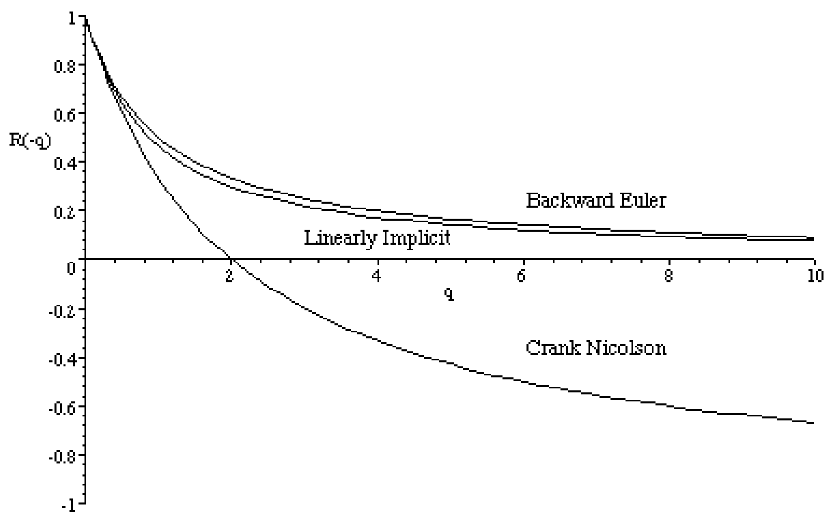
$$v^{n+1} = R(q)v^n,\ q = k\lambda$$

$$R(q) = \frac{P(q)}{Q(q)} = \frac{1 + 2\phi q + (\phi^2 + \phi - \frac{1}{4})q^2}{(1 - (\frac{1}{2} - \phi)q)^2}$$

A--acceptable if $|R(q)| < 1$ whenever $Re(q) < 0$ and *L--acceptable* if, in addition, $|R(q)| \to 0$ as $Re(q) \to -\infty$

L--stability$\Rightarrow \phi^2 + \phi - \frac{1}{4} = 0$ or $\phi = -\frac{1}{2} \pm \frac{\sqrt{2}}{2}$

Positivity preserving $\Rightarrow \phi = -\frac{1}{2} - \frac{\sqrt{2}}{2}$



Stability functions

European Put Option

Transformed and dimensionless form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + (k-1)\frac{\partial u}{\partial x} - ku, a \leq x \leq b, t_0 \leq t \leq t_f,$$

$$k = \frac{r}{(\sigma^2/2)}, r = 0.065, \sigma = 0.8, a = \log(\tfrac{2}{5})$$

$$b = \log(\tfrac{7}{5}), t_0 = 0, t_f = 5,$$

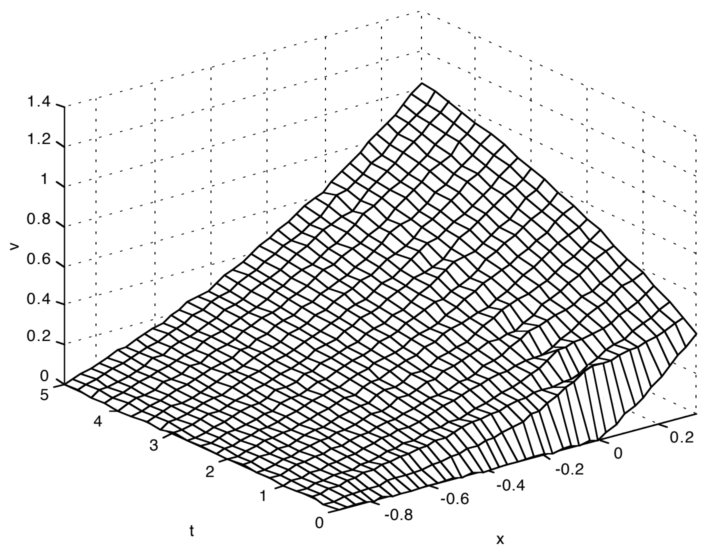
initial conditions

$$u(x,0) \;=\; \max(\exp(x) - 1, 0)$$

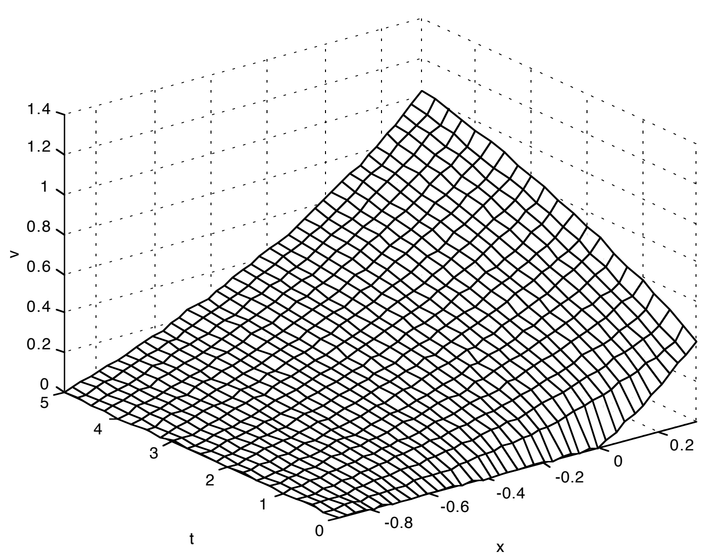
boundary conditions

$$u(a,t) = 0, \qquad u(b,t) = \frac{7 - 5\exp(-kt)}{5}$$

$$\Delta x = 0.1313, \Delta t = 0.25$$



European Option: Crank Nicolson method



European Option: Linearly implicit method

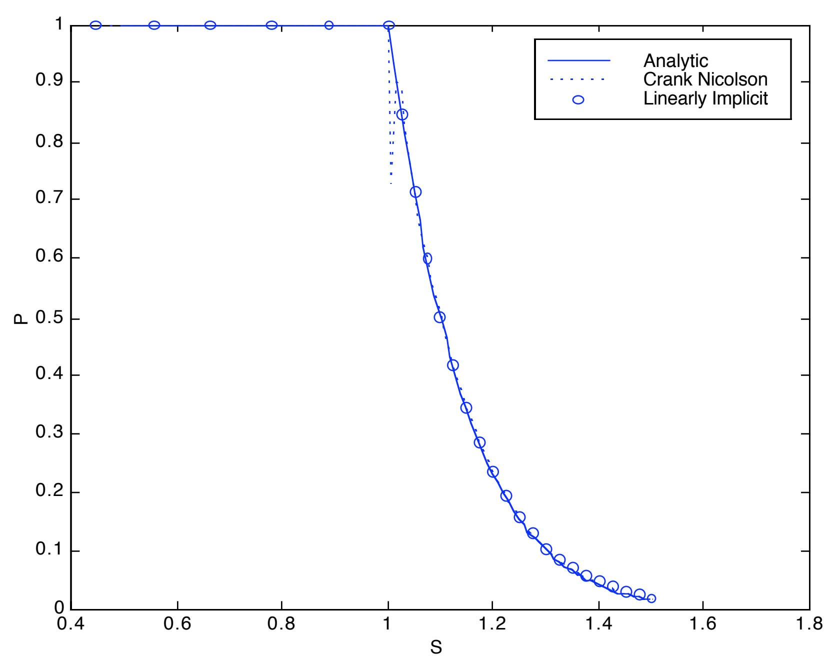
Digital Options

payoff is nonsmooth

optimal exercise boundary is at $\overline{S}(t) = E$

American put option

$$
\begin{aligned}
\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP &= 0, \\
S > E, 0 \le t < T & \\
P(S,T) &= \begin{cases} 1 & S \le E \\ 0 & S > E \end{cases}, \\
P(E,t) &= 1, \\
\lim_{S\to\infty} P(S,t) &= 0, \\
P(S,t) &= 1, 0 \le S \le E
\end{aligned}
$$



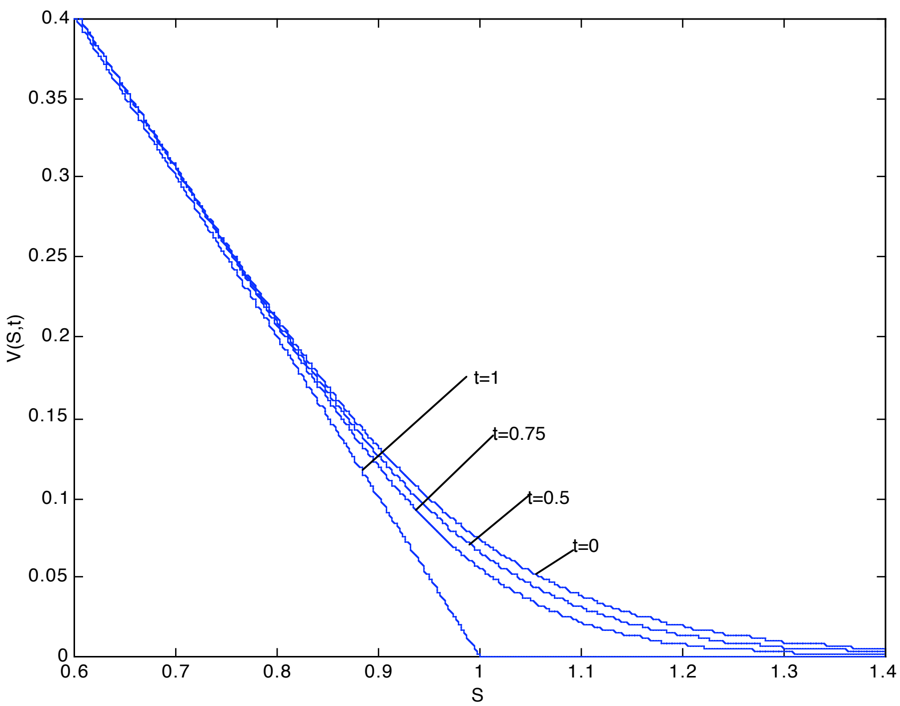
American Digital Option

American put option

$$\frac{\partial V_\epsilon}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_\epsilon}{\partial S^2} + rS\frac{\partial V_\epsilon}{\partial S} - rV_\epsilon \qquad S \in [0, S_\infty]$$
$$+\frac{\epsilon C}{V_\epsilon + \epsilon - q(S)} \quad = \quad 0,$$

$$V_\epsilon(S,T) = \max(E - S, 0),$$
$$V_\epsilon(0,t) = E,$$
$$V_\epsilon(S_\infty, t) = 0,$$

$$C \geq rE, q(S) = E - S$$



American Option: Linearly implicit method
$r = 0.1, \sigma = 2, E = 1, T = 1, S_\infty = 2$
$\Delta S = 10^{-3}, \Delta t = 5 \times 10^{-4}, \epsilon = 10^{-2}$

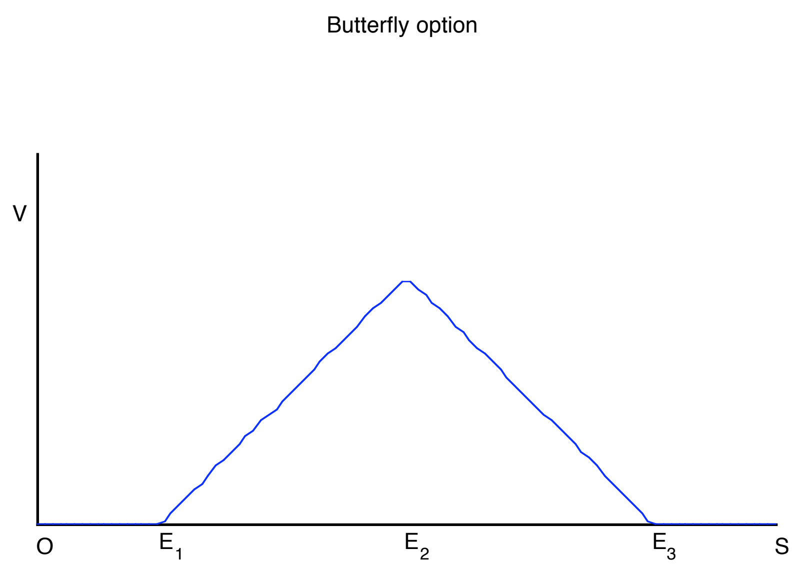
Butterfly option

- strike prices: E_1, E_2, E_3

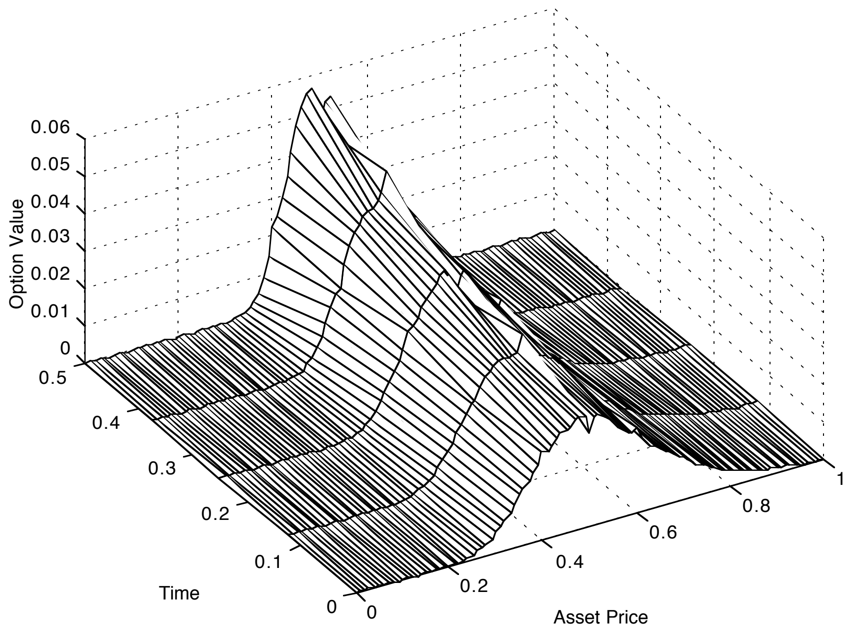
$$E_1 < E_2 < E_3, E_2 = \frac{(E_1 + E_3)}{2}$$

- payoff

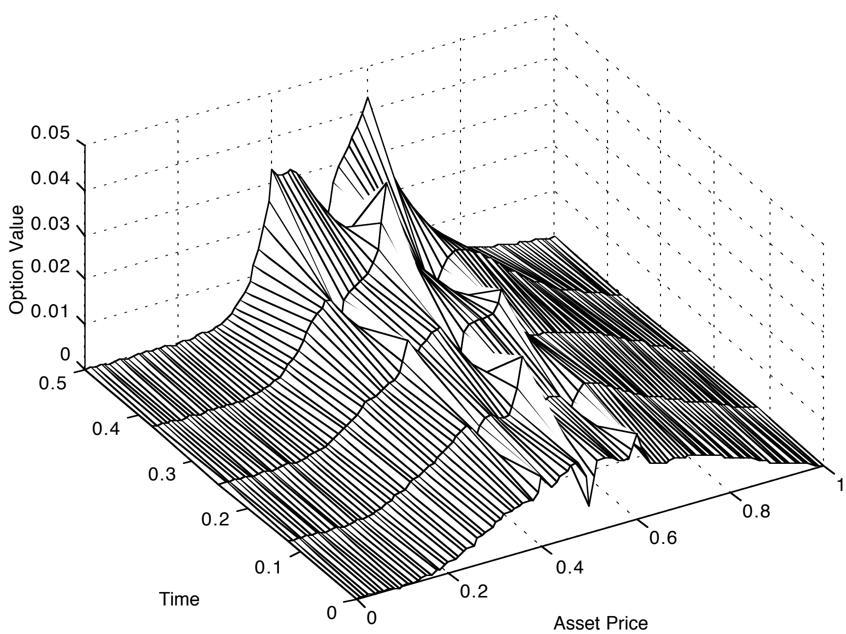
$$V(S,T) = \max(S - E_1, 0) - 2\max(S - E_2, 0) + \max(S - E_3, 0)$$



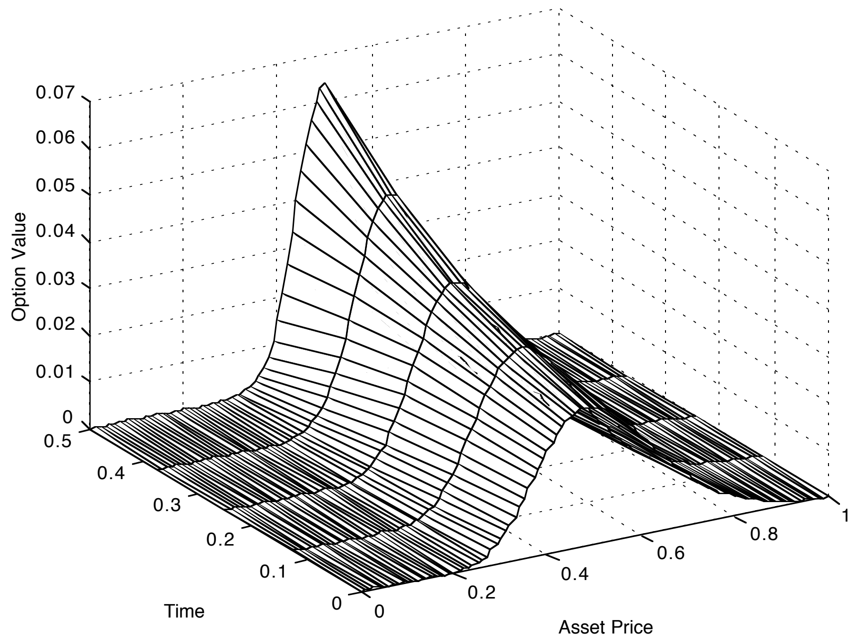
Hull (2003) Prentice Hall



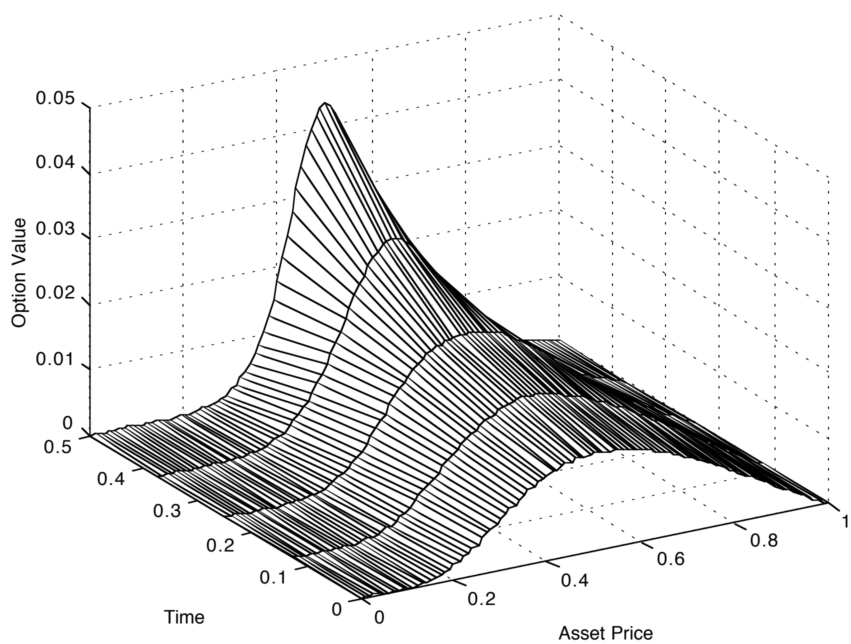
Butterfly option: Crank-Nicolson method: $\sigma = .35$



Butterfly option: Crank-Nicolson method: $\sigma = .7$



Butterfly option: Linearly implicit method: $\sigma = .35$



Butterfly option: Linearly implicit method: $\sigma = .7$

Two–asset option

- Assets: S_1, S_2 with correlation ρ

- Brownian motion:

$$\begin{aligned} dS_1 &= \mu_1 S_1 dt + \sigma_1 S_1 dW^{(1)} \\ dS_2 &= \mu_2 S_2 dt + \sigma_2 S_2 dW^{(2)} \\ E(dW^{(1)} dW^{(2)}) &= \rho dt \end{aligned}$$

- Ito–Lemma and no–arbitrage

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + rS_1 \frac{\partial V}{\partial S_1} - rV$$
$$+\frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + rS_2 \frac{\partial V}{\partial S_2} + \frac{1}{2}\rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} = 0$$

- Payoff for a call

$$(f(S_1(T), S_2(T) - E)^+$$

$$f = \begin{cases} \alpha S_1 + (1 - \alpha S_2 \\ \mathsf{max}(S_1, S_2) \\ \mathsf{min}(S_1, S_2) \end{cases}$$

general n--factor model

- Assets: $S_i, i = 1 \ldots n$ with correlation $\rho_{i,j}$

- Dividend: $\delta_i, i = 1 \ldots n$

- PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{n} \rho_{i,j} \sigma_i \sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n} (r - \delta_i) S_i \frac{\partial V}{\partial S_i} - rV = 0$$

Two-asset American Put Option

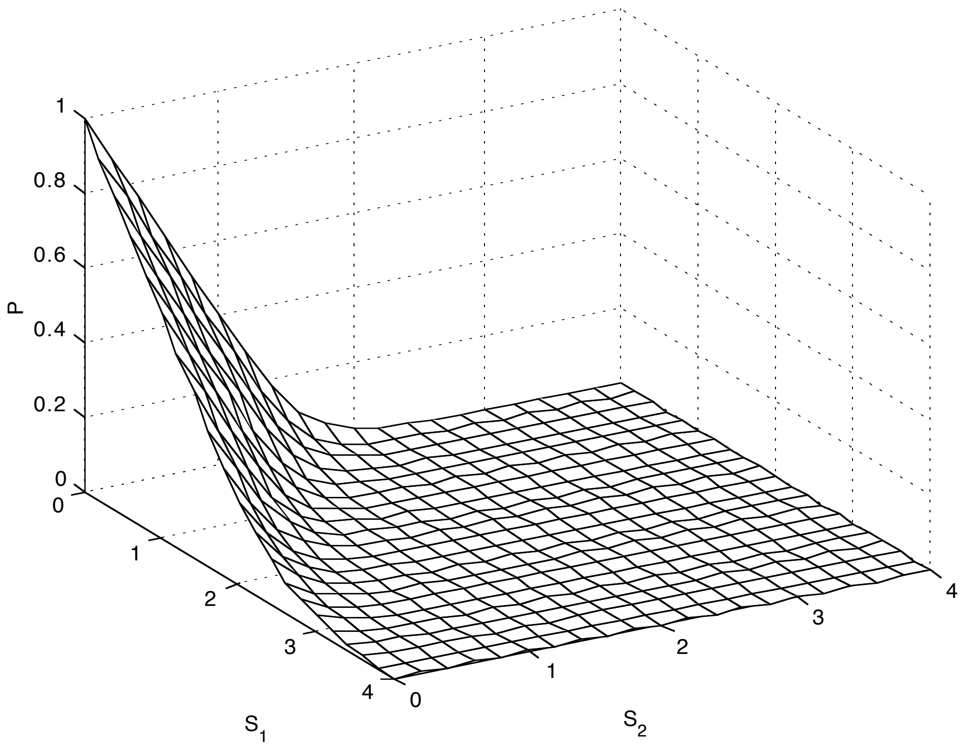
$$\frac{\partial P}{\partial t}+\frac{1}{2}\sigma_1^2S_1^2\frac{\partial^2 P}{\partial S_1^2}+\frac{1}{2}\sigma_2^2S_2^2\frac{\partial^2 P}{\partial S_2^2}+\frac{1}{2}\rho\sigma_1\sigma_2S_1S_2\frac{\partial^2 P}{\partial S_1\partial S_2}$$
$$+(r-\delta_1)S_1\frac{\partial P}{\partial S_1}+(r-\delta_2)S_2\frac{\partial P}{\partial S_2}-rP+\frac{\epsilon C}{P+\epsilon-q}=0,$$
$$S_1,S_2\geq 0, t\in[0,T)$$

$$\begin{aligned}
P(S_1,S_2,T) &= \phi(S_1,S_2), S_1,S_2\geq 0,\\
P(S_1,0,t) &= g_1(S_1,t), S_1\geq 0, t\in[0,T]\\
P(0,S_2,t) &= g_2(S_2,t), S_2\geq 0, t\in[0,T]\\
\lim_{S_2\to\infty} P(S_1,S_2,t) &= G_1(S_1,t), S_1\geq 0, t\in[0,T]\\
\lim_{S_1\to\infty} P(S_1,S_2,t) &= G_2(S_2,t), S_2\geq 0, t\in[0,T]
\end{aligned}$$

where

$$\begin{aligned}
q(S_1,S_2) &= E-(\alpha_1S_1+\alpha_2S_2)\\
\phi(S_1,S_2) &= \max(q(S_1,S_2),0)
\end{aligned}$$

$$\begin{aligned}
r &= 0.1,\\
\sigma_1 &= 0.2, \sigma_2=0.3\\
\alpha_1 &= 0.6, \alpha_2=0.4\\
\delta_1 &= 0.05, \delta_2=0.01\\
E &= 1.0, T=1.0
\end{aligned}$$



Present value of two--asset American put option
$t = 0, \epsilon = .01, \rho = 0.5, S_1(\infty) = 4, S_2(\infty) = 4$
$\Delta S = 0.2, \Delta t = 0.4$

Fasshauer, Khaliq, Voss (2004): mesh free
Wade, Khaliq, et. al.(2004): finite difference
Khaliq, Voss, Kazmi (2005): linearly implicit

4. Future work and challenges

(Paul Wilmott on Quantitative Finance (2000) Wiley)

- Jump diffusion models

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rS\frac{\partial V}{\partial S} - rV$$
$$+\lambda E[V(JS,t) - V(S,t)] - \lambda S \frac{\partial V}{\partial S} E[(J-1)] = 0$$

where

$$E[x] = \int x P(J) dJ$$

and $P(J)$ is the pdf for the jump size J

- Transaction costs

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \kappa\sigma S^2 \sqrt{\frac{2}{\pi \delta t}} \left| \frac{\partial^2 V}{\partial S^2} \right|$$
$$+rS\frac{\partial V}{\partial S} - rV = 0$$

- Multiscale stochastic volatility

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \frac{\partial^2 V}{\partial \sigma^2}$$
$$+ rS\frac{\partial V}{\partial S} + (p - \lambda q)\frac{\partial V}{\partial \sigma} - rV = 0$$

- Stochastic optimal control models

$$\begin{aligned}
dW &= ((w(\mu - r) + r)W - C)dt + w\sigma W dX \\
J(W,t) &= \max_{C,w} E|_t \left[\int_t^T \exp -\rho\tau U(C(\tau))d\tau + B(W)\right] \\
0 &= \max_{C,w}(\exp -\rho\tau U(C) + \frac{\partial J}{\partial t} \\
&\quad + ((w(\mu - r) + r)W - C)\frac{\partial J}{\partial W} \\
&\quad + \frac{1}{2} w^2 \sigma^2 W^2 \frac{\partial^2 V}{\partial W^2})
\end{aligned}$$